ON THE VALUE DISTRIBUTION OF $f^l(f^{(k)})^n$

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Abstract

Quantitative estimations on the value distribution of $f^l(f^{(k)})^n$ are studied in this paper. As a result of this, some known results are improved.

1. Introduction

Let f denote a transcendental meromorphic function, and the usual symbols: $T(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, $m(r, f)$, $S(r, f)$ of Nevanlinna value distribution theory see, e.g. [7], are used throughout the paper.

A complex value a is said to be a Picard value of f , if and only if, $f(z) - a$ has at most finitely many zeros. W. K. Hayman [8] conjectured that the only possible Picard value of $f^n f'$ is zero, and he himself proved the case when $n \geq 3$ in [10], and left the cases of $n = 1, 2$. Later on, Mues [11] proved the case for $n = 2$, and afterwards Clunie [4] proved the case for $n = 1$ when f is entire. An affirmative answer to the case when f is meromorphic and $n = 1$ is yet to be resolved. Since then a stream of studies on questions of possible Picard values of differential polynomials of f has been launched, and many related results have been obtained, see e.g. [1]–[5] and [10]–[19]. In 1981, Steinmetz [12] proved:

THEOREM A. *Let f be a transcendental meromorphic function in the plane. If $n_0, \dots, n_k \geq 0$, $n_0 \geq 2$, $n_1 + \dots + n_k \geq 1$ and $\phi = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k} - 1$, then*

$$\limsup_{r \rightarrow \infty} \frac{\bar{N}(r, 1/\phi)}{T(r, \phi)} > 0.$$

In 1982, Doeringer [5] proved the following:

THEOREM B. *Let f be a transcendental meromorphic function, $Q(f)$ and $P(f)$ be two non-zero differential polynomials and $\phi = f^n Q(f) + P(f)$. Then for any natural number n with $n \geq 3 + \gamma_P$ (γ_P : the weight of $P(f)$),*

$$\limsup_{r \rightarrow \infty} \frac{\bar{N}(r, 1/\phi)}{T(r, \phi)} > 0.$$

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Remark. Particularly, when $n \geq 3$, $\gamma_p = 0$ (i.e. $P(f)$ is a non-zero small function of f), we can derive from this that

$$\limsup_{r \rightarrow \infty} \frac{\bar{N}(r, 1/\phi)}{T(r, f)} \geq \frac{1}{2}.$$

Thus Theorem B, in some sense, is an improvement and extension of Theorem A. Furthermore, it gives a quantitative estimation of the number of zeros of ϕ . Recently, in [17], the following result has been obtained:

THEOREM C. *Let f be a transcendental entire function and n, k be two non-negative integers with $n \geq 2$, $k \geq 0$. Then $f'f^{(k)n}$ assumes every non-zero finite value infinitely many times.*

Hence it is natural to ask:

PROBLEM D. Does Theorem C also hold when f is a transcendental meromorphic function?

In this paper, as an attempt in resolving Problem D, we obtain some quantitative estimations on the zeros of $f^l(f^{(k)})^n - 1$ with $l=1, 2$ and $k, n \geq 2$ by argument different from that of Theorems B and C, and give an affirmative answer to problem D when $k \geq 0$ and $n > 9e+1$.

2. Main Result

THEOREM. *Let f be a transcendental meromorphic function in the plane and $F = f^l(f^{(k)})^n - 1$ with l, k and n being three positive integers and $l \leq 2$.*

(i) If $l=1$ and $n > 9e+1$, then there exists some constant $K > 1$, a set $M(K)$ of upper logarithmic density at most $\delta(K) = \min((2e^{K-1}-1)^{-1}, (1+e(K-1))\exp(e(1-K)))$, and a set D of finite linear measure such that $n-9eK-1 \geq \varepsilon > 0$ and

$$(\varepsilon - o(1))T(r, f^{(k)}) \leq 2\bar{N}\left(r, \frac{1}{F}\right), \quad r \in E(K)$$

where $E(K) = [0, \infty) \setminus M(K) \cup D$ (note that for $K > 1$, $m(E(K)) = \infty$),

Particularly F assumes zero infinitely often.

(ii) If $l=2$ and $k, n \geq 2$, then

$$\left(\frac{1}{2} - \eta\right)T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f),$$

for every $0 < \eta < (1/2)$.

Remark. The case of $l \geq 3$ has been taken care of by Theorem B.

In order to prove the above theorem, we shall make use of the following lemmas.

LEMMA 1 (Frank [6]). *If k is a positive integer and $\varepsilon > 0$, then*

$$k\bar{N}(r, f) \leq N\left(r, \frac{1}{f^{(k)}}\right) + (1 + \varepsilon)N(r, f) + S(r, f).$$

LEMMA 2 (Hayman-Miles [9]). *Suppose that $f(z)$ is a transcendental meromorphic function and K , a constant, > 1 . Then there exists a set $M(K)$ with upper logarithmic density at most*

$$\delta(K) = \min((2e^{K-1} - 1)^{-1}, (1 + e(K-1) \exp(e(1-K))))),$$

such that for every positive integer k ,

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eK, \quad r \notin M(K).$$

In particular when r is large and $r \notin M(K)$, then

$$-3eKT(r, f^{(k)}) \leq -(1 - o(1))T(r, f). \quad (2.1)$$

LEMMA 3. *If $F = f(f^{(k)})^n - 1$; $n, k \geq 1$ and $g = (F'/F)$, then*

$$m(r, H(z)) = S(r, f) - S(r, f^{(k)}),$$

where $H(z) = nf^{(k+1)} + (f'/f)f^{(k)} - gf^{(k)}$.

Proof. From $g = (F'/F)$, we have

$$f(f^{(k)})^{n-1} \left(nf^{(k+1)} + \frac{f'}{f} f^{(k)} - g f^{(k)} \right) = -g. \quad (2.2)$$

Let E_1 be the set of θ in $[0, 2\pi]$ for which $|f(re^{i\theta})| < 1$, E_2 be the set of θ in $[0, 2\pi]$ for which $|f(re^{i\theta})| \geq 1$ and $|f^{(k)}(re^{i\theta})| \geq 1$, and E_3 be the set $[0, 2\pi] \setminus E_1 \cup E_2$. It is easy to see that for $\theta \in E_1$ and $z = re^{i\theta}$,

$$\begin{aligned} \log^+ |H(z)| &\leq \log^+ |g(z)| + \log^+ \left| \frac{f'(z)}{f(z)} \right| + 2 \log^+ \left| \frac{f^{(k)}(z)}{f(z)} \right| \\ &\quad + n \log^+ \left| \frac{f^{(k+1)}(z)}{f(z)} \right| + O(1), \end{aligned} \quad (2.3)$$

and

$$\log^+ |H(z)| \leq n \log^+ \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| + \log^+ \left| \frac{f'(z)}{f(z)} \right| + \log^+ |g(z)| + O(1), \quad (2.4)$$

for $\theta \in E_3$ and $z = re^{i\theta}$. Now, by (2.2), we have

$$\log^+ |H(z)| \leq \log^+ |g(z)| + \log^+ |f(z)f^{(k)}(z)|^{-1} + O(1),$$

so for $\theta \in E_2$ and $z = re^{i\theta}$,

$$\log^+ |H(z)| \leq \log^+ |g(z)| + O(1). \quad (2.5)$$

Hence

$$\begin{aligned} \int_0^{2\pi} \log^+ |H(re^{i\theta})| d\theta &= \int_{E_1} \log^+ |H(re^{i\theta})| d\theta + \int_{E_2} \log^+ |H(re^{i\theta})| d\theta \\ &\quad + \int_{E_3} \log^+ |H(re^{i\theta})| d\theta. \end{aligned}$$

From this, (2.3), (2.4) and (2.5), and by the well-known lemma on the logarithmic derivative [7], we have

$$\begin{aligned} m(r, H(z)) &\leq O(m(r, g)) + O\left(m\left(r, \frac{f^{(k)}}{f}\right)\right) + O\left(m\left(r, \frac{f^{(k+1)}}{f}\right)\right) \\ &\quad + O\left(m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right)\right) + O(1) \\ &\leq S(r, f) + S(r, f^{(k)}), \end{aligned}$$

and this also completes the proof of the lemma.

LEMMA 4. With F , g and H as defined in Lemma 3, we have

$$(n-1)T(r, f^{(k)}) \leq 3T(r, f) + 2\bar{N}\left(r, \frac{1}{F}\right) + S(r, f) + S(r, f^{(k)}).$$

Proof. From the definition of $H(z)$, we see immediately that the possible poles of $H(z)$ occur only at the poles of f and the zeros of F and f . Now note that g can have only simple poles, and hence by (2.2), it is easily verified that any pole of f , say z_0 , cannot be a pole of $H(z)$. Consequently

$$N(r, H) \leq \bar{N}\left(r, \frac{1}{F}\right) + N_1\left(r, \frac{1}{f}\right).$$

Now combining this fact with lemma 3, (2.2), and by Nevanlinna's first fundamental theorem [7], we have

$$(n-1)T(r, f^{(k)}) \leq T(r, f) + T(r, g) + T(r, H).$$

Consequently

$$\begin{aligned} (n-1)T(r, f^{(k)}) &\leq T(r, f) + \bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{F}\right) + N_1\left(r, \frac{1}{f}\right) \\ &\quad + S(r, f) + S(r, f^{(k)}), \end{aligned}$$

where $N_1(r, 1/f)$ denotes the counting function corresponding the simple zeros

of f . It follows that

$$(n-1)T(r, f^{(k)}) \leq 3T(r, f) + 2\bar{N}\left(r, \frac{1}{F}\right) + S(r, f) + S(r, f^{(k)}).$$

3. The Proof of the Theorem

Proof of (i). Choose $K > 1$ such that $n-1-9eK > 0$. We note that $S(r, f) = o(1)T(r, f)$ outside a finite linear measure A , and $S(r, f^{(k)}) = o(1)T(r, f^{(k)})$ outside a finite linear measure B , also by Lemma 2

$$T(r, f) = O(1)T(r, f^{(k)}), \quad r \notin M(K).$$

Therefore $S(r, f) + S(r, f^{(k)}) = o(1)T(r, f^{(k)})$ on $E(K) = [0, \infty) \setminus M(K) \cup A \cup B$. Now from lemma 4,

$$(n-1)T(r, f^{(k)}) - 3T(r, f) \leq 2\bar{N}\left(r, \frac{1}{F}\right) + o(1)T(r, f^{(k)}), \quad r \in E(K).$$

Thus, it follows from this and (2.1) of Lemma 2 that

$$(n-1-9eK-o(1))T(r, f^{(k)}) \leq 2\bar{N}\left(r, \frac{1}{F}\right), \quad r \in E(K)$$

and, therefore, there exists a positive constant $\varepsilon \leq n-1-9eK$ such that

$$(\varepsilon - o(1))T(r, f^{(k)}) \leq 2\bar{N}\left(r, \frac{1}{F}\right), \quad r \in E(K). \quad (3.1)$$

Now if F assumes zero finite times, then

$$\frac{\bar{N}(r, (1/F))}{T(r, f^{(k)})} \rightarrow 0, \quad \text{as } r \rightarrow \infty, r \in E(K)$$

and it follows that $\varepsilon < 0$ from (3.1). This is a contradiction and hence, F must have infinitely many zeros.

Proof of (ii). Let $F = f^2(f^{(k)})^n - 1$ with $k, n \geq 2$. Then we have

$$\frac{1}{f^{n+2}} = \left(\frac{f^{(k)}}{f}\right)^n - \frac{F'F}{f^{n+2}F'}.$$

It follows that

$$m\left(r, \frac{1}{f^{n+2}}\right) \leq m\left(r, \frac{F}{F'}\right) + m\left(r, \frac{F'}{f^{n+2}}\right) + S(r, f).$$

Note that F' is a homogenous differential polynomial in f of degree $n+2$. Hence $m(r, (F'/f^{n+2})) = S(r, f)$ and

$$m\left(r, \frac{1}{f^{n+2}}\right) \leq m\left(r, \frac{F}{F'}\right) + S(r, f).$$

Thus, again by the first fundamental theorem, the above can be rewritten as

$$m\left(r, \frac{1}{f^{n+2}}\right) \leq m\left(r, \frac{F'}{F}\right) + N\left(r, \frac{F'}{F}\right) - N\left(r, \frac{F}{F'}\right) + S(r, f).$$

Note $m(r, F'/F) = S(r, F) = S(r, f)$. The above yields

$$m\left(r, \frac{1}{f^{n+2}}\right) \leq N\left(r, \frac{F'}{F}\right) - N\left(r, \frac{F}{F'}\right) + S(r, f).$$

Consequently

$$m\left(r, \frac{1}{f^{n+2}}\right) \leq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) - N\left(r, \frac{1}{F'}\right) + S(r, f). \quad (3.2)$$

On the other hand, from $F' = f(f^{(k)})^{n-1}(nf f^{(k+1)} + 2f' f^{(k)})$, we have

$$N\left(r, \frac{1}{f}\right) + (n-1)N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{F'}\right). \quad (3.3)$$

Substituting (3.3) into (3.2) and then adding $N(r, (1/f^{n+2}))$ to both sides of (3.2), we get

$$\begin{aligned} T\left(r, \frac{1}{f^{n+2}}\right) &\leq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) + N\left(r, \frac{1}{f^{n+2}}\right) - N\left(r, \frac{1}{f}\right) \\ &\quad - (n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned}$$

Hence

$$\begin{aligned} (n+2)T(r, f) &\leq N\left(r, \frac{1}{F}\right) + \bar{N}(r, f) + (n+1)N\left(r, \frac{1}{f}\right) \\ &\quad - (n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned}$$

Now by combining this and lemma 1, we have for any given $0 < \varepsilon < 1$,

$$\begin{aligned} (n+2)T(r, f) &\leq N\left(r, \frac{1}{F}\right) + \frac{1}{2}N\left(r, \frac{1}{f^{(k)}}\right) + \frac{1+\varepsilon}{2}N(r, f) + (n+1)N\left(r, \frac{1}{f}\right) \\ &\quad - (n-1)N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned}$$

Thus,

$$(n+2)T(r, f) \leq N\left(r, \frac{1}{F}\right) + \left(n+1 + \frac{1+\varepsilon}{2}\right)T(r, f) + S(r, f),$$

which leads to

$$\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)T(r, f) \leq N\left(r, \frac{1}{F}\right) + S(r, f).$$

This also completes the proof of the theorem.

Remark. It is easily seen the results can be extended to the case when the value 1 for F is replaced by any non-zero small function of f .

REFERENCES

- [1] C.T. CHUANG, On differential polynomials, Analysis of One Complex Variable, World Scientific Publishing Co., 1987, pp. 12-32.
- [2] C.T. CHUANG AND X.H. HUA, On the growth of differential polynomials, Sci. Sinica, vol. 33, no. 9 (1990), pp. 1025-1033.
- [3] C.T. CHUANG AND X.H. HUA, On a conjecture of Hayman, Acta Mathematica Sinica, New series, vol. 7, no. 2 (1991), pp. 119-126.
- [4] J. CLUNIE, On integral and meromorphic functions, J. London Math. Soc., 37 (1962), pp. 17-27.
- [5] W. DOERINGER, Exceptional values of differential polynomials, Pacific J. Math., vol. 98, no. 1 (1982), pp. 55-62.
- [6] G. FRANK AND G. WEISSENBORN, Rational deficient functions of meromorphic functions, Bull. London Math. Soc., 18 (1986), pp. 29-33.
- [7] W.K. HAYMAN, Meromorphic Functions, Oxford, 1975.
- [8] W.K. HAYMAN, Research Problems in Function Theory, London Univ. Press, 1967.
- [9] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, Complex Variables, 12 (1989), pp. 245.
- [10] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, Ann. of Math., 70 (1959), pp. 9-42.
- [11] E. MUES, Über ein Problem von Hayman, Math. Z., 164 (1979), pp. 239-259.
- [12] N. STEINMETZ, Über die Nullstellen von Differential Polynomen, Math. Z., 176 (1981), pp. 255-264.
- [13] L.R. SONS, Deficiencies of monomials, Math. Z., 111 (1969), pp. 53-68.
- [14] C.C. YANG, On deficiencies of differential polynomials, Math. Z., 116 (1970), pp. 197-204.
- [15] C.C. YANG, On deficiencies of differential polynomials II, Math. Z., 125 (1972), pp. 107-112.
- [16] C.C. YANG AND H.X. YI, Some quantitative estimations of the zeros of differential polynomials, Science in China Ser. A, vol. 23, no. 1 (1980), pp. 8-20 (in Chinese).
- [17] C.C. YANG, L. YANG AND Y.F. WANG, On zeros of $f(f^{(k)})^n - 1$, to appear.
- [18] H.X. YI, On differential polynomials, Acta Mathematica Sinica, vol. 33, no. 3 1990 (in Chinese).
- [19] H.X. YI, On the value distribution of ff' , Kexue Tongbao, 10 (1989), pp. 727-730 (in Chinese).

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