A UNIQUENESS THEOREM FOR MEROMORPHIC FUNCTIONS WHOSE N -TH DERIVATIVES SHARE THE SAME 1-POINTS

By

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1. Introduction

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and c be a point in the extended complex plane. If $f(z) - c$ and $g(z) - c$ have exactly the same zero set (i.e., counting multiplicities), then we say f and g share c -points and denote this relationship by $f = c \rightleftharpoons g = c$ [4]. In this note, it is assumed that the reader is familiar with the standard notations of Nevanlinna's theory that can be found, for instance, in [3].

In 1976, M. Ozawa [4] proved the following result:

Theorem A *Let f and g be two nonconstant entire functions. If $f = 1 \rightleftharpoons g = 1$ with $\delta(0, f) > 0$ and 0 being a Picard-exceptional value of g , then either $f \equiv g$ or $fg \equiv 1$.*

H. Yi [8] improved the above result by deriving the following:

Theorem B *Let f and g be two meromorphic functions satisfying $\delta(\infty, f) = \delta(\infty, g) = 1$. If $f = 1 \rightleftharpoons g = 1$ with $\delta(0, f) + \delta(0, g) > 1$, then either $f \equiv g$ or $fg \equiv 1$.*

C. Yang [6] studied the possible relationships when two entire functions together with their first derivatives share, respectively, 0-points. There he asked: What can be said if two nonconstant entire functions f and g satisfy $f = 0 \rightleftharpoons g = 0$ and $f' = 1 \rightleftharpoons g' = 1$?

As an attempt to solve the above question, K. Shibazaki [5] obtained the following result.

Theorem C *Let f and g be two entire functions of finite order. If $f' = 1 \rightleftharpoons g' = 1$ with $\delta(0, f) > 0$ and 0 is a Picard-exceptional value of g , then either $f \equiv g$ or $f'g' \equiv 1$.*

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Recently Yi–Yang [7] extended the above result by considering meromorphic functions, in general, and obtained the following:

Theorem D *Let f and g be two meromorphic functions satisfying $\delta(\infty, f) = \delta(\infty, g) = 1$. If $f' = 1 \rightleftharpoons g' = 1$ and $\delta(0, f) + \delta(0, g) > 1$, then either $f \equiv g$ or $f'g' \equiv 1$.*

It has been asked in [7] whether a similar conclusion holds if the first derivatives f' and g' in Theorem D are replaced by their n th derivatives $f^{(n)}$ and $g^{(n)}$? In this note, we prove a theorem which provides an affirmative answer to this question and includes Theorems A–D as corollaries.

2. Main results and preliminary lemmas

Theorem 1 *Let f and g be two meromorphic functions with $\Theta(\infty, f) = \Theta(\infty, g) = 1$. If $f^{(n)} = 1 \rightleftharpoons g^{(n)} = 1$ and $\delta(0, f) + \delta(0, g) > 1$, then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

In the sequel, we shall use I and E to denote an arbitrary set of values in $(0, \infty)$ of infinite or finite linear measure, respectively, not necessarily the same each occurrence.

Lemma 1 *Let F be a nonconstant meromorphic function satisfying $\Theta(\infty, F) = 1$. Then, for any nonnegative integer n ,*

$$T(r, F^{(n)}) \leq (1 + o(1))T(r, F) \quad (r \notin E).$$

Proof From $\Theta(\infty, F) = 1$, it follows that

$$\bar{N}(r, F) = o(T(r, F)).$$

Hence

$$\begin{aligned} T(r, F^{(n)}) &= m(r, F^{(n)}) + N(r, F^{(n)}) \\ &\leq m(r, F) + m\left(r, \frac{F^{(n)}}{F}\right) + N(r, F) + n\bar{N}(r, F) \\ &\leq (1 + o(1))T(r, F) \quad (r \notin E). \end{aligned}$$

Lemma 2 *Let F be as in Lemma 1. Then*

$$N\left(r, \frac{1}{F^{(n)}}\right) \leq (1 - \delta(0, F))T(r, F^{(n)}) + o(T(r, F)) \quad (r \notin E).$$

Proof From

$$\begin{aligned} m\left(r, \frac{1}{F}\right) &\leq m\left(r, \frac{1}{F^{(n)}}\right) + m\left(r, \frac{F^{(n)}}{F}\right) \\ &= m\left(r, \frac{1}{F^{(n)}}\right) + o(T(r, F)) \quad (r \notin E) \end{aligned}$$

and the definition of $\delta(0, F)$, we have

$$(\delta(0, F) + o(1))T(r, F) \leq T(r, F^{(n)}) - N\left(r, \frac{1}{F^{(n)}}\right) + o(T(r, F)) \quad (r \notin E).$$

By Lemma 1, we obtain

$$(\delta(0, F) + o(1))T(r, F^{(n)}) \leq T(r, F^{(n)}) - N\left(r, \frac{1}{F^{(n)}}\right) + o(T(r, F)) \quad (r \notin E).$$

This yields

$$N\left(r, \frac{1}{F^{(n)}}\right) \leq (1 - \delta(0, F) + o(1))T(r, F^{(n)}) + o(T(r, F)) \quad (r \notin E).$$

Lemma 3 *Let $F(z)$ be as in Lemma 1 with $\delta(0, F) > 0$. Then*

$$T(r, F) \leq \left(\frac{1}{\delta(0, F)} + o(1)\right)T(r, F^{(n)}) \quad (r \notin E).$$

Proof As in the proof of Lemma 2, we have

$$(\delta(0, F) + o(1))T(r, F) \leq T(r, F^{(n)}) - N\left(r, \frac{1}{F^{(n)}}\right) + o(T(r, F)) \quad (r \notin E).$$

This assertion follows immediately from the above by noting that

$$N(r, 1/F^{(n)}) \geq 0$$

Lemma 4 *Assume that f and g satisfy the hypothesis of Theorem 1. Then*

$$T(r, f) = O(1)T(r, g) \quad (r \notin E)$$

and

$$T(r, g) = O(1)T(r, f) \quad (r \notin E).$$

Proof According to the Milloux inequality [3],

$$(1) \quad T(r, f) < \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(n)} - 1}\right) + o(T(r, f)) \quad (r \notin E).$$

We note that under the assumptions

$$(2) \quad \bar{N}(r, f) = o(T(r, f))$$

and

$$(3) \quad N\left(r, \frac{1}{f}\right) \leq (1 - \delta(0, f) + o(1))T(r, f),$$

$$\bar{N}\left(r, \frac{1}{f^{(n)} - 1}\right) = \bar{N}\left(r, \frac{1}{g^{(n)} - 1}\right) \leq T(r, g^{(n)}) + O(1).$$

Also, by Lemma 1

$$\bar{N}\left(r, \frac{1}{f^{(n)} - 1}\right) \leq (1 + o(1))T(r, g) \quad (r \notin E).$$

On combining this with (1), (2), and (3) we can deduce

$$(\delta(0, f) + o(1))T(r, f) \leq (1 + o(1))T(r, g) \quad (r \notin E).$$

Since $\delta(0, f) + \delta(0, g) > 1$, it follows that $\delta(0, f) > 0$. Thus, from the above, we have

$$T(r, f) = O(1)T(r, g) \quad (r \notin E).$$

Similarly, we have

$$T(r, g) = O(1)T(r, f) \quad (r \notin E).$$

Lemma 5 *Let f_1 and f_2 be two nonconstant meromorphic functions and c_i ($i = 1, 2, 3$) be three nonzero constants. If $c_1 f_1 + c_2 f_2 \equiv c_3$ holds, then*

$$T(r, f_1) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_1) + o(T(r, f_1)) \quad (r \notin E).$$

Proof It follows from Nevanlinna's Second Fundamental Theorem that

$$\begin{aligned} T(r, f_1) &< \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_1 - c_3/c_1}\right) + \bar{N}(r, f_1) + o(T(r, f_1)) \\ &= \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_1) + o(T(r, f_1)) \quad (r \notin E). \end{aligned}$$

Lemma 6 ([2], p.70). *Let f_i ($i = 1, 2, \dots, n$) be n linearly independent meromorphic functions satisfying*

$$\sum_{i=1}^n f_i \equiv 1.$$

Then, for $j = 1, 2, \dots, n$,

$$\begin{aligned} T(r, f_j) &< \sum_{i=1}^n N\left(r, \frac{1}{f_i}\right) + N(r, f_j) + N(r, D) - \sum_{i=1}^n N(r, f_i) - N\left(r, \frac{1}{D}\right) \\ &+ O(1)(\log r + \log T(r)) \quad (r \notin E), \end{aligned}$$

where

$$D = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

$$T(r) = \max_{1 \leq i \leq n} [T(r, f_i)].$$

Let $F(z)$ be a meromorphic function and k a positive integer. We denote by $n_k(r, 1/F)$ the number of zeros of F within $|z| \leq r$, zeros of multiplicities $< k$ being counted multiply, while zero of multiplicities $\geq k$ are counted k times. The associated counting function will be denoted by $N_k(r, 1/F)$ [1].

Lemma 7 *Let f_i ($i = 1, 2, 3$) be three meromorphic functions with $f_1 \not\equiv \text{constant}$. Suppose that $\sum_{i=1}^3 f_i \equiv 1$ and that*

$$\sum_{i=1}^3 \tilde{N}(r, f_i) = o(T(r)) \quad (r \notin E)$$

and

$$(4) \quad \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) \leq (\lambda + o(1))T(r) \quad (r \in I),$$

where

$$T(r) = \max_{1 \leq i \leq 3} \{T(r, f_i)\},$$

and λ is positive < 1 . Then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

Proof First of all, if either f_2 or f_3 is a constant function $\not\equiv 1$, then by Lemma 5 this will lead to a contradiction. Thus we may assume that both f_2 and f_3 are

nonconstant functions. Now if f_i ($i = 1, 2, 3$) are linear independent, then, by Lemma 6,

$$(5) \quad T(r, f_3) < \sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) + N(r, D) - \sum_{i=1}^2 N(r, f_i) - N\left(r, \frac{1}{D}\right) + o(T(r)) \quad (r \notin E),$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}.$$

Notice that $\sum_{i=1}^3 f_i \equiv 1$; we have

$$D = \begin{vmatrix} f_1 & f_2 & 1 \\ f_1' & f_2' & 0 \\ f_1'' & f_2'' & 0 \end{vmatrix} = \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}$$

and hence

$$(6) \quad N(r, D) - \sum_{i=1}^2 N(r, f_i) \leq 2 \sum_{i=1}^2 \bar{N}(r, f_i) = o(T(r)) \quad (r \notin E).$$

Let z_0 be a zero of some f_i of multiplicity p (≥ 2). Then z_0 is a zero of multiplicity $p-1$ and $p-2$ of $f_i'(z)$ and $f_i''(z)$, respectively. Thus z_0 is a zero of D with multiplicity at least $p-2$. It follows that

$$(7) \quad \sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) - N\left(r, \frac{1}{D}\right) \leq \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right).$$

On combining inequalities (5), (6) and (7), we have

$$T(r, f_3) < \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) + o(T(r)) \quad (r \notin E).$$

We have a similar inequality for f_2 and f_1 . Hence, for $i \equiv 1, 2, 3$,

$$T(r, f_i) < \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) + o(T(r)) \quad (r \notin E).$$

It follows that

$$T(r) < \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) + o(T(r)) \quad (r \notin E).$$

This and (4) yield

$$T(r) \leq (\lambda + o(1))T(r) \quad (r \in I),$$

a contradiction, since $\lambda < 1$. Thus we conclude that f_1, f_2 and f_3 are linearly dependent, i.e., there exist constants c_i ($i = 1, 2, 3$) not all equal to zero such that

$$(8) \quad \sum_{i=1}^3 c_i f_i \equiv 0.$$

If $c_1 \neq 0$, then it follows from (8) and the assumption $\sum_{i=1}^3 f_i \equiv 1$ that

$$(1 - c_2/c_1)f_2 - (1 - c_3/c_1)f_3 \equiv 1,$$

and so

$$T(r, f_i) = (1 + o(1))T(r), \quad i = 1, 2, 3.$$

However, this is impossible by Lemma 5. Thus $c_1 = 0$. Then, again from (8) and the assumption $\sum_{i=1}^3 f_i \equiv 1$,

$$f_1 + (1 - c_3/c_2)f_3 \equiv 1.$$

As before, a contradiction will result. This also completes the proof.

Lemma 8 *Assume that f and g satisfy the conditions of Theorem 1 and, in addition to that, $f^{(n)} \equiv g^{(n)}$. Then $f \equiv g$.*

Proof Suppose that $f \not\equiv g$. From $f^{(n)} \equiv g^{(n)}$, we have

$$f \equiv g + p,$$

where p is a polynomial ($\neq 0$) of degree $< n$. Evidently, by $\Theta(\infty, f) = 1$ and $\delta(0, f) > 0$, f has to be a transcendental function. Thus $T(r, p) = o(T(r, f))$ and hence

$$T(r, g) = (1 + o(1))T(r, f).$$

According to Nevanlinna's Second Fundamental Theorem for three deficient functions [3, p.47], we have

$$\begin{aligned}
 T(r, f) &< \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(\frac{1}{f-p}\right) + o(T(r, f)) \quad (r \notin E) \\
 &= \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + o(T(r, f)) \\
 &\leq (1 - \delta(0, f))T(r, f) + (1 - \delta(0, g))T(r, g) \\
 &\quad + o(T(r, f)) + o(T(r, g)) \quad (r \notin E) \\
 &= (2 - \delta(0, f) - \delta(0, g) + o(1))T(r, f) \quad (r \notin E).
 \end{aligned}$$

This is impossible, since $2 - \delta(0, f) - \delta(0, g) < 1$. Thus we must have $f \equiv g$.

3. Proof of Theorem 1

It is easily verified now that, under the hypothesis of the theorem, either

$$T(r, f^{(n)}) \leq T(r, g^{(n)}) \quad (r \in I)$$

or

$$T(r, g^{(n)}) \geq T(r, f^{(n)}) \quad (r \in I)$$

is valid. We may assume without loss of generality that the latter holds.

Because $\delta(0, f) + \delta(0, g) > 1$, it follows that $\delta(0, f) > 0$ and $\delta(0, g) > 0$. And, by the assumption $\Theta(\infty, f) = \Theta(\infty, g) = 1$ with Lemma 3, we have

$$T(r, f) = O(1)T(r, f^{(n)}) \quad (r \notin E).$$

On the other hand, by virtue of Lemma 4,

$$T(r, g) = O(1)T(r, f^{(n)}) \quad (r \notin E).$$

Thus

$$\bar{N}(r, f) = o(T(r, f^{(n)})) \quad (r \notin E)$$

and

$$\bar{N}(r, g) = o(T(r, f^{(n)})) \quad (r \notin E).$$

Now set

$$h = \frac{f^{(n)} - 1}{g^{(n)} - 1}.$$

By $f^{(n)} = 1 \Leftrightarrow g^{(n)} = 1$, we obtain $\bar{N}(r, h) \leq \bar{N}(r, f)$ and $\bar{N}(r, 1/h) \leq \bar{N}(r, g)$. Thus

$$\bar{N}(r, h) + \bar{N}\left(r, \frac{1}{h}\right) = o(T(r, f^{(n)})) \quad (r \notin E).$$

By letting $f_1 \equiv f^{(n)}$, $f_2 \equiv h$, and $f_3 \equiv -hg^{(n)}$ we have $\sum_{i=1}^3 f_i \equiv 1$ and

$$\sum_{i=1}^3 \bar{N}(r, f_i) \leq \bar{N}(r, f) + \bar{N}(r, g) + 2\bar{N}(r, h) = o(T(r)) \quad (r \notin E)$$

and

$$\begin{aligned} \sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) &\leq N\left(r, \frac{1}{f^{(n)}}\right) + N\left(r, \frac{1}{g^{(n)}}\right) + 4\bar{N}\left(r, \frac{1}{h}\right) \\ &= N\left(r, \frac{1}{f^{(n)}}\right) + N\left(r, \frac{1}{g^{(n)}}\right) + o(T(r)) \quad (r \notin E), \end{aligned}$$

where

$$T(r) = \max_{1 \leq i \leq 3} \{T(r, f_i)\}.$$

Lemma 2 yields

$$\begin{aligned} N\left(r, \frac{1}{f^{(n)}}\right) &\leq (1 - \delta(0, f) + o(1))T(r) \quad (r \notin E), \\ N\left(r, \frac{1}{g^{(n)}}\right) &\leq (1 - \delta(0, g) + o(1))T(r, g^{(n)}) \\ &\leq (1 - \delta(0, g) + o(1))T(r) \quad (r \in I). \end{aligned}$$

Thus

$$\sum_{i=1}^3 N_2\left(r, \frac{1}{f_i}\right) \leq (\lambda + o(1))T(r) \quad (r \in I),$$

where $\lambda = 2 - \delta(0, f) - \delta(0, g) < 1$. An application of Lemma 7 yields either $f_2 \equiv 1$ or $f_3 \equiv 1$.

If $f_2 \equiv 1$, then $f^{(n)} \equiv g^{(n)}$ and hence $f \equiv g$ by Lemma 8. If $f_3 \equiv 1$, then $g^{(n)} \equiv -1/h$, $f^{(n)} \equiv -h$, and so $f^{(n)}g^{(n)} \equiv 1$. The theorem is thus proven.

4. Concluding remarks

(i) Theorem 1 serves as a generalization of Theorems A through D.

(ii) Let

$$f(z) = -\frac{1}{2^n}e^{2z} + e^z, \quad g(z) = -\frac{(-1)^n}{2^n}e^{-2z} + (-1)^ne^{-z} \quad (n \text{ a non-negative integer}).$$

Then it is easily verified that $\Theta(\infty, f) = \Theta(\infty, g) = 1$, $f^{(n)} = 1 \Leftrightarrow g^{(n)} = 1$ and

$$\delta(0, f) + \delta(0, g) = 1.$$

Moreover, $f \not\equiv g$ and $f^{(n)}g^{(n)} \not\equiv 1$. This illustrates that the condition $\delta(0, f) + \delta(0, g) > 1$ of Theorem 1 is necessary.

Finally, by exactly the same arguments as in proving Theorem 1, one can obtain the following more general result.

Theorem 2 *Let f and g be two nonconstant meromorphic functions and $a(z)$ be a meromorphic function ($\neq 0$) satisfying*

$$T(r, a(z)) = o(1) \min\{T(r, f), T(r, g)\}.$$

Suppose that $f^{(n)}(z) - a(z) = 0 \Leftrightarrow g^{(n)}(z) - a(z) = 0$; $n \geq 0$ and $\delta(0, f) + \delta(0, g) > 1$. Then either $f \equiv g$ or $f^{(n)}(z)g^{(n)}(z) \equiv a^2(z)$.

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