ON THE FIX-POINTS OF COMPOSITE MEROMORPHIC FUNCTIONS AND GENERALIZATIONS

By

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1. Introduction

Let us begin with introducing the following fundamental notations and definition. Let $f(z)$ be a transcendental function meromorphic in the complex plane. Throughout this paper, we denote by $\rho(f)$, $\lambda(f)$ and $\sigma(f)$ the order, lower order of $f(z)$, and exponent of convergence for its zeros, respectively and by E and F sets on the positive real axis with, respectively, finite linear measure and finite logarithmic measure, not necessarily the same at each occurrence. And we denote by $S(r, f)$ a quantity $S(r, f) = o(T(r, f))$, as $r \rightarrow \infty$, possibly outside a set of finite linear measure. We say that a meromorphic function $\gamma(z)$ is a small meromorphic function with respect to $f(z)$, provided that $T(r, \gamma) = S(r, f)$. Let $g(z)$ be a transcendental entire function, or else meromorphic if $f(z)$ is a rational function. We require the following

Definition The small function-points (resp. polynomial-points) of $f(g(z))$ are the zeros of $f(g(z)) - \gamma(z)$, where $\gamma(z)$ is a small function with respect to $g(z)$ (resp. a non-constant polynomial).

Finally, we assume that the reader is familiar with Nevanlinna's theory of meromorphic functions and its standard notations, see [27].

Fatou [16, 17] and Julia [29, 30] were the first to make thorough studies of the geometric properties of the fix-points of iteration[†] of rational functions. Since their time, similar investigations for iteration of transcendental entire functions or meromorphic functions of special forms have been pursued by many mathematicians, including P. C. Rosenbloom [38], H. Töpfer [42], I. N. Baker [1, 2, 3], C. T. Chuang [12], W. Bergweiler [5, 6], J. K. Langley [33] and A. Eremenko and M. Yu. Lyubich [15], to name just a few. However, only after the development of value distribution theory of meromorphic functions by Nevanlinna in 1925 [35] did quantitative estimation on the fix-points of iterations of entire functions become possible. In 1952, Rosenbloom [38] extended the study of the existence of the fix-points of iteration of one function to that of a composition of two functions and

* The research of the first author was partially supported by a U.P.G.C. grant, Hong Kong, and the second author by a research grant from Tsing Hua University, China.

obtained some quantitative estimates on the fix-points of a composite function of the form $P(f(z))$ in terms of the growth of $f(z)$, where $P(z)$ is a nonlinear polynomial and $f(z)$ a transcendental entire function. Then, in 1970, in connection with the study of factorization theory, Gross raised the following conjecture ([20], [22], [24]):

Conjecture 1 Let $f(z)$ and $g(z)$ be two transcendental entire functions. Then $f(g(z))$ must have an infinite number of fix-points.

Furthermore, in [20], the following question was raised:

Question 1 Do there exist a transcendental meromorphic function $f(z)$ and a transcendental entire function $g(z)$ such that $f(g(z))$ has a finite number of fix-points?

In the sequel, unless specified otherwise, $f(z)$ and $g(z)$ will denote transcendental entire functions. When the order $\rho(f(g))$ of $f(g)$ is finite, Conjecture 1 was proved by Gross–Yang [25], Propokovich [37], and Goldstein [19], more or less simultaneously but by quite different arguments. Later on, utilizing the classical Wiman–Valiron theory [43], Bergweiler [5] gave a complete answer to Conjecture 1 and subsequently settled Question 1 in [8]. Earlier in [5], it had been shown that for any non-constant polynomial $P(z)$, $n(r)$, the number of zeros of $f(g) - P$ in $|z| \leq r$, tends to infinity as $r \rightarrow \infty$ as does the counting function $N(r)$ corresponding to $n(r)$. Two further questions were raised there: (a) What are lower bounds for $n(r)$ or $N(r)$? (b) Is the number of small function-points of $f(g(z))$ always unbounded? It was Langley [33] who established a more precise estimate for $N(r)$, viz, $N(r) \neq o(T(r, f(g)))$, when $f(g)$ has finite order and $P(z)$ is allowed to be a non-constant rational function. This is a good answer to Question (a) in the special case. Katajamäki, Kinnunen and Laine [31] showed that the answer to Question (b) is affirmative when both $f(z)$ and $g(z)$ are entire functions of finite order. Actually, it was proved in [31] that $T(r, \prod) \neq o(T(r, g))$, where $\prod(z)$ denotes the canonical product formed with zeros of $f(g) - \gamma$. The present authors [47] established that $N(r, 1/(f(g) - P)) > dT(r^{1/3}, g)$ as $r \rightarrow \infty$, $r \in I$, where d is a positive number less than 1, and I a set of infinite logarithmic measure. Let $\prod_1(z)$ and $\prod_2(z)$ denote the canonical products formed with the zeros and poles of $f(g) - \gamma$, respectively. It was noted in [31] that if $f(z)$ is meromorphic, then for $\prod(z) = \prod_1(z)/\prod_2(z)$ the conclusion $T(r, \prod) \neq o(T(r, g))$ remains true, but it does not imply that $\prod_1(z)$ must have infinitely many zeros. Thus an additional question (Question (c)) was asked in [31], namely: will the conclusion of Question (b) hold if $f(z)$ and $\gamma(z)$ are meromorphic functions? Bergweiler–Yang [11] were able to settle Question (c) when the order of $f(g)$ is finite and $\gamma(z)$ a non-constant rational function. This result was further generalized in [32], where it was shown that the exponent of convergence for zeros of $f(g) - \gamma$ is $\geq \lambda(g)$, (the lower order of $g(z)$)

when $f(z)$ has finite order and $g(z)$ has finite lower order. In [45], it was conjectured that $N(r, 1/(f(g) - \gamma)) \neq o(T(r, g))$, for any non-constant meromorphic function $\gamma(z)$ satisfying $T(r, \gamma) = S(r, g)$. Zheng [49] settled the conjecture when $f(z)$ is a meromorphic function with finite order, $g(z)$ is an entire function with finite lower order and $\gamma(z)$ is a small function with respect to $g(z)$. Actually, it was shown in [49] that, under the previous assumptions, we have $T(r, g) = o(\bar{N}(r, 1/(f(g) - \gamma)))$; this is still true under the assumption that $\log T(r, f) < (\log r)/(\log \log r)^{1+\epsilon}$, $\epsilon > 0$, without any restriction on the growth of $g(z)$. All the above results have led us to propose the following

Conjecture 2 [45] *Let $f(z)$ be a transcendental meromorphic function, $g(z)$ a transcendental entire function and $\gamma(z)$ a non-constant meromorphic function satisfying $T(r, \gamma) = S(r, g)$. Then*

$$N\left(r, \frac{1}{f(g) - \gamma}\right) \neq o(T(r, f(g))).$$

In this paper, based on the techniques developed by Bergweiler, Propokovich, Steinmetz and the authors, we shall present findings which are improvements over many of the results mentioned above. Some of these results are interesting in their own right and should be useful in further research on value distribution theory. The paper is divided into 4 sections. Section 2 consists of the preliminary results or known lemmas on canonical products that will be employed in this paper. Section 3 deals mainly with sharp bounds or refined estimations of the small function-points of $R(f)$, and more generally, zeros of functional polynomial $P(z, f)$, where $R(z)$ is a rational function. In Section 4, the growth properties of polynomial-points of the composition of entire functions are studied. In each section, related results will be mentioned briefly. Our ultimate goal is the confirmation of Conjecture 2. However, it seems that some new techniques and deeper analysis are needed for a complete confirmation or negation of Conjecture 2.

2. Growth properties of canonical products

It has been established in the conclusion of [47], and will be done here again, that there exists a close relation between the number of the fix-points or polynomial-points of composite entire functions and the growth of the canonical product formed with those points. In this paper, we need an estimate for canonical products derived by Gol'dberg [18] in 1972 and (for infinite order) by Bergweiler [9] in 1992. The latter will be further developed here. We refer the reader to [9] for a survey of related results.

Let $\{z_j\}$ be a sequence of complex numbers with $0 < |z_1| \leq |z_2| \leq \dots$ and $\lim_{j \rightarrow \infty} |z_j| = \infty$. Define

$$(1) \quad \mu = \limsup_{r \rightarrow \infty} \frac{\log \log n(r)}{\log \log r},$$

where $n(r)$ is the number of $\{z_j\}$ in $|z| \leq r$. It is obvious that if

$$(2) \quad \limsup_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = +\infty,$$

then $1 \leq \mu \leq \infty$.

First of all, let us state the result of Gol'dberg [18, Theorem 1], which will be used to prove Theorem 4.2.

Theorem 2.1 *Given $\{z_j\}$ as above, there exists an entire function $\alpha(z)$ with $\{z_j\}$ as its sole zeros such that*

$$(3) \quad \log \log M(r, \alpha) = o(N(r)), \quad r \notin F,$$

where $N(r)$ is the counting function corresponding to $n(r)$.

In order to make the discussion of canonical product of infinite order clearer, we shall give a detailed proof of our results concerning this subject. To this end, we need the following estimation of the Weierstraß primary factors due to J. H. E. Cohn [14].

Lemma 2.1 *Let $E(z, p)$ be a Weierstraß primary factor, i.e., $E(z, p) = (1 - z) \exp(z + z^2/2 + \dots + z^p/p)$. Then*

$$(4) \quad \log |E(z, p)| < \frac{3}{2}|z|^p \quad (p \geq 1, |z| \geq 1, z \neq 1)$$

and

$$(5) \quad \log |E(z, p)| < \frac{3}{2}|z|^{p+1} \quad (p \geq 1, |z| \leq 1, z \neq 1).$$

Actually, Cohn [14] proved more, but the above special case suffices for our purpose.

Theorem 2.2 *Given the sequence $\{z_j\}$ whose moduli tend monotonically to infinity and satisfy (2), then there exists an entire function $\alpha(z)$ whose zeros are precisely the z_j 's such that*

(I)

$$(6) \quad \liminf_{r \rightarrow \infty} \frac{\log M(r, \alpha)}{n(r)} \leq K,$$

where $K = 3\mu/(4(\mu - 1))$ if $1 < \mu < +\infty$ and $K = 3/4$ if $\mu = \infty$;

(II) for $\mu = 1$

$$(7) \quad \liminf_{r \rightarrow \infty} \frac{\log M(r, \alpha)}{n(r) \log n(r)} \leq \frac{3(e - 1)}{2}.$$

Proof Here we only need to prove Case (II), for the proof of Case (I) can be found in Bergweiler [9].

From (2), there exists an unbounded subset I of the positive real set such that for $r \in I$,

$$(8) \quad \frac{\log n(r)}{\log r} = \sup \left\{ \frac{\log n(t)}{\log t}; 1 \leq t \leq r \right\}.$$

It is easy to see that

$$(9) \quad \lim_{\substack{r \rightarrow \infty \\ r \in I}} \frac{\log n(r)}{\log r} = +\infty.$$

Set $r_j = |z_j|$. We now define sequences $\{R_m\} \subset I$, $\{T_m\}$ and $\{p_j\}$ of non-negative integers by recursion. First choose R_1 and T_1 with $\max\{1, r_1\} \leq R_1 < T_1$ and then define $p_j = 0$ for $1 \leq j \leq n(T_1)$. Assume that R_k , T_k have been defined for $1 \leq k \leq m - 1$, $m \geq 2$ and p_j for $1 \leq j \leq n(T_{m-1})$. Then we define R_m satisfying

(i) $R_m > T_{m-1}$, $R_m \in I$

and

(ii)

$$\sum_{r_j \leq T_{m-1}} \left(\frac{R_m}{r_j} \right)^{p_j} \leq n(T_{m-1}) R_m^{q_{m-1}} \leq \frac{1}{m} n(R_m),$$

where $q_{m-1} = \max\{p_j; 1 \leq j \leq n(T_{m-1})\}$, which we can do by (9). We define, for $T_{m-1} < r_j \leq R_m$,

$$p_j = \left[\left(1 + \frac{1}{\log n(R_m)} \right) \frac{\log j}{\log r_j} \right].$$

And we choose T_m satisfying

(iii) $R_m^{n(R_m)} \leq T_m$

and

(iv) $[3 \log R_m]^{2 \log n(R_m)} \leq n(T_m)$,

and for $R_m < r_j \leq T_m$, define p_j such that

$$p_j \geq \left\lceil 2 \frac{\log j}{\log r_j} \right\rceil$$

and

$$(10) \quad \sum_{R_m < r_j \leq T_m} \left(\frac{R_m}{r_j} \right)^{p_j+1} \leq 1.$$

Now we define the Weierstraß product $\alpha(z)$ by

$$\alpha(z) = \prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, p_j\right).$$

Then for $m \geq 2$,

$$(11) \quad \begin{aligned} \log M(R_m, \alpha) &\leq \left(\sum_{r_1 \leq r_j \leq T_1} + \sum_{T_1 < r_j \leq R_m} + \sum_{R_m < r_j \leq T_m} + \sum_{T_m < r_j < \infty} \right) \\ &\quad \times \log M(R_m, E(z/z_j, p_j)) \\ &= S_1 + S_2 + S_3 + S_4 \text{ (say).} \end{aligned}$$

We shall estimate each of the S_i 's. For S_1 , we have

$$(12) \quad \begin{aligned} S_1 &\leq \sum_{r_1 \leq r_j \leq T_1} \log \left(1 + \frac{R_m}{r_j} \right) \leq \sum_{r_1 \leq r_j \leq T_1} \frac{R_m}{r_j} \\ &= \frac{1}{r_1} n(T_1) R_m = o(n(R_m)). \end{aligned}$$

From $R_m \in I$, we have for $T_1 < t \leq R_m$

$$\begin{aligned} \left(\frac{R_m}{t} \right)^{\left(1 + \frac{1}{\log n(R_m)}\right) \frac{\log n(t)}{\log t}} &= \exp \left[\left(1 + \frac{1}{\log n(R_m)}\right) \left(\log R_m \cdot \frac{\log n(t)}{\log t} - \log n(t) \right) \right] \\ &\leq \exp \left[\left(1 + \frac{1}{\log n(R_m)}\right) \log \frac{n(R_m)}{n(t)} \right] \\ &= \left(\frac{n(R_m)}{n(t)} \right)^{1 - \frac{1}{\log n(R_m)}}, \end{aligned}$$

and therefore from (4) in Lemma 2.1 and (ii) in the definition of R_m ,

$$\begin{aligned}
 S_2 &\leq \frac{3}{2} \sum_{T_1 < r_j \leq R_m} \left(\frac{R_m}{r_j} \right)^{p_j} \\
 &\leq \frac{3}{2m} n(R_m) + \frac{3}{2} \int_{T_{m-1}}^{R_m} \left(\frac{R_m}{t} \right)^{\left(1 + \frac{1}{\log n(R_m)}\right) \frac{\log n(t)}{\log t}} dn(t) \\
 (13) \quad &= \frac{3e}{2} n(R_m) \int_{T_{m-1}}^{R_m} n(t)^{-1 - \frac{1}{\log n(R_m)}} dn(t) + o(n(R_m)) \\
 &= \frac{3e}{2} n(R_m) \log n(R_m) \left[n(T_{m-1})^{-\frac{1}{\log n(R_m)}} - \frac{1}{e} \right] + o(n(R_m)) \\
 &= \frac{3(e-1)}{2} n(R_m) \log n(R_m) + o(n(R_m)).
 \end{aligned}$$

It is clear that from (5) in Lemma 2.1 and (10) in the definition of p_j , we have $S_3 < \frac{3}{2}$. Finally, in order to estimate S_4 , from (iii), we observe for $T_m \leq t < +\infty$ that

$$\begin{aligned}
 \left(1 + \frac{1}{\log n(R_m)}\right) \left(\frac{\log R_m}{\log t} - 1\right) &\leq \left(1 + \frac{1}{\log n(R_m)}\right) \left(\frac{\log R_m}{\log T_m} - 1\right) \\
 &\leq \left(1 + \frac{1}{\log n(R_m)}\right) \left(\frac{1}{n(R_m)} - 1\right) \\
 &\leq -1 - \frac{1}{2 \log n(R_m)},
 \end{aligned}$$

which, from (5) in Lemma 2.1 and the choice of T_m in (iv), we deduce that

$$\begin{aligned}
 S_4 &< \frac{3}{2} \int_{T_m}^{\infty} \left(\frac{R_m}{t} \right)^{\left(1 + \frac{1}{\log n(R_m)}\right) \frac{\log n(t)}{\log t}} dn(t) \\
 &= \frac{3}{2} \int_{T_m}^{\infty} n(t)^{\left(1 + \frac{1}{\log n(R_m)}\right) \left(\frac{\log R_m}{\log t} - 1\right)} dn(t) \\
 &\leq \frac{3}{2} \int_{T_m}^{\infty} n(t)^{-1 - \frac{1}{2 \log n(R_m)}} dn(t) \\
 &= 3n(T_m)^{-\frac{1}{2 \log n(R_m)}} \log n(R_m) \leq 1.
 \end{aligned}$$

Obviously, the above estimates of S_1 , S_2 , S_3 and S_4 lead to (7), and Theorem 2.2 follows. $\square$

By the following result, we describe the sharpness of (7) in Theorem 2.2 in a certain sense. Some ideas in the proof of Theorem 2.3 come from Bergweiler [10].

Theorem 2.3 *There exist a sequence $\{z_k\}$ satisfying*

$$(14) \quad \lim_{k \rightarrow \infty} |z_k| = \infty,$$

$$(15) \quad \liminf_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = 0, \quad \limsup_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = \infty,$$

$$(16) \quad \lim_{r \rightarrow \infty} \frac{\log \log n(r)}{\log \log r} = 1$$

and a set J with upper logarithmic density one such that for all entire functions $f(z)$ having the zero sequence $\{z_k\}$, the inequality

$$(17) \quad \log n(r)[\log \log n(r)]^2 \leq \log \log M(r, f), \quad r \in J$$

holds. Furthermore, we have

$$n(r)[\log n(r)]^2 \leq \log M(r, f), \quad r \in J.$$

Proof First we define by recursion the sequence $\{r_j\}$ of positive real numbers

$$r_1 > e^e, \quad \log r_{j+1} = \log r_j (\log \log r_{j+1})^4$$

satisfying $r_j \rightarrow \infty$ and

$$\log \log r_{j+1} \sim \log \log r_j,$$

as $j \rightarrow \infty$, and the sequence $\{p_j\}$ of integers

$$p_0 = 0, \quad p_j = \left[r_{j+1}^{1/\log \log r_j} \right]$$

such that $p_j = o(p_{j+1})$ as $j \rightarrow \infty$. Next we define the sequence $\{z_k\}$ by $z_k = r_j$ for $p_{j-1} + 1 \leq k \leq p_j$ and $j \geq 1$. Then for $r_j \leq r < r_{j+1}$, we have

$$n(r) = p_j \leq r_{j+1}^{1/\log \log r_j},$$

and furthermore we easily deduce (14), (15) and (16). Choose the sequence $\{r'_j\}$ by

$$r'_j = r_j^{1/\log \log r_{j-1}}, \quad r_{j-1} < r'_j < r_j$$

and define the set

$$J = \bigcup_{j=1}^{\infty} [r'_j, r_j).$$

Then J has upper logarithmic density 1. Actually, we have

$$\begin{aligned} \frac{1}{\log r_j} \int_{J \cap [1, r_j]} \frac{dt}{t} &\geq \frac{1}{\log r_j} \sum_{s=1}^j \log \frac{r_s}{r'_s} \\ &\geq \frac{\log r_j - \log r'_j}{\log r_j} = 1 - \frac{\log r'_j}{\log r_j} \\ &= 1 - \frac{1}{\log \log r_{j-1}} \rightarrow 1, \end{aligned}$$

as $j \rightarrow \infty$.

Below we want to prove that $\{z_k\}$ and J satisfy the desired properties. Let $f(z)$ be a transcendental entire function with the zero sequence $\{z_k\}$. Since $f(z) \neq 0$ in $|z| \leq e$, we can expand in $|z| \leq e$

$$\log f(z) = \sum_{m=0}^{\infty} a_m z^m,$$

with $a_m \rightarrow 0$, as $m \rightarrow \infty$. And therefore

$$\begin{aligned} c_m(r) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \log |f(re^{i\theta})| d\theta \\ &= \frac{1}{2} a_m r^m + \frac{1}{2m} \sum_{s=1}^j (p_s - p_{s-1}) \left(\left(\frac{r}{r_s} \right)^m - \left(\frac{r_s}{r} \right)^m \right). \end{aligned}$$

For $r'_{j+1} \leq r < r_{j+1}$, we have

$$(18) \quad |c_m(r)| \geq \frac{1}{2m} \sum_{s=1}^j (p_s - p_{s-1}) \left(\left(\frac{r}{r_s} \right)^m - \left(\frac{r_s}{r} \right)^m \right) - \frac{1}{2} |a_m| r^m$$

$$(19) \quad \geq \frac{1}{2m} (p_j - p_{j-1}) \left(\frac{r}{r_j} \right)^m - \frac{1}{2m} p_j - \frac{1}{2} |a_m| r^m.$$

Set

$$m = [\log \log r_{j+1}]^2.$$

Then

$$\frac{p_j}{m} \frac{1}{r_j^m} \rightarrow \infty$$

and

$$\left(\frac{r}{r_j}\right)^m \geq \left(\frac{r'_{j+1}}{r_j}\right)^m \rightarrow \infty.$$

Then from (19), we have

$$\log M(r, f) \geq |c_m(r)| \geq \frac{1}{3m} p_j \left(\frac{r}{r_j}\right)^m \geq (r'_{j+1})^m$$

and

$$\begin{aligned} \log \log M(r, f) &\geq [\log \log r_{j+1}]^2 \log r'_{j+1} \\ &\geq \log n(r) [\log \log n(r)]^2. \end{aligned}$$

Theorem 2.3 follows. □

3. Composition of rational and meromorphic functions

Let $f(z)$ be a transcendental meromorphic function and denote by $\Omega(f)$ the field generated by all the small meromorphic functions with respect to $f(z)$.

Consider the polynomial in w

$$(20) \quad P(z, w) := a_m(z)w^m + \cdots + a_1(z)w + a_0(z),$$

where $a_j(z)$ ($0 \leq j \leq m$) with $a_m(z) \not\equiv 0$ are entire functions, having no common zeros, at least one of which is not a constant. Let $w(z)$ be an m -valued algebroid function defined by $P(z, w) = 0$. By $n(r, w)$ and $n_x(r, w)$ we denote the number of poles and branch points of $w(z)$ in $|z| \leq r$, respectively. Specifically, assume that in some neighborhood of $z = z_0$ ($|z_0| \leq r$), $w(z)$ has the following expansion:

$$w(z) = b_{-\tau}(z - z_0)^{-\tau/\lambda} + b_{-\tau+1}(z - z_0)^{(-\tau+1)/\lambda} + \cdots.$$

Then we contribute the ramification index $\lambda - 1$ into $n_x(r, w)$ and if $\tau > 0$, contribute τ into $n(r, w)$. As in Nevanlinna theory of meromorphic functions, we define, following Selberg [39], the counting function for an m -valued algebroid function $w(z)$ by

$$N(r, w) = \int_0^r \frac{n(t, w) - n(0, w)}{t} dt + n(0, w) \log r,$$

$$N_x(r, w) = \int_0^r \frac{n_x(t, w) - n_x(0, w)}{t} dt + n_x(0, w) \log r,$$

$$N^*(r, w) = \frac{1}{m}N(r, w) \quad \text{and} \quad N_x^*(r, w) = \frac{1}{m}N_x(r, w).$$

Actually, $N(r, w) = N(r, 1/a_m)$. Now through the branch points of $w(z)$, we draw a non-self-intersecting open polygon Γ_w to cut the complex plane in such a way that: (i) any circle of finite radius intersects no more than finitely many components of Γ_w ; (ii) Γ_w contains no poles of $w(z)$ except its branch points; and (iii) $C \setminus \Gamma_w$ is a connected domain, where $w(z)$ has m one-valued analytic branches w_j ($1 \leq j \leq m$). We define the proximity function of $w(z)$ as follows:

$$m(r, w) = \sum_{j=1}^m \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w_j(re^{i\theta})| d\theta.$$

Set

$$T(r, w) = m(r, w) + N(r, w).$$

As shown by Selberg [39] and Valiron [44] the first fundamental theorem holds for algebroid functions as well as the following:

Lemma 3.1 *Let $w(z)$ be an m -valued algebroid function determined by (20). Then*

$$(21) \quad T(r, w) = O(\max\{T(r, a_j); 0 \leq j \leq m\});$$

$$(22) \quad m\left(r, \frac{w^{(k)}}{w}\right) = O(\log r T(r, w)), \quad r \notin E;$$

$$(23) \quad N_x(r, w) \leq 2(m-1)T(r, w) + O(1);$$

$$(24) \quad N(r, w^{(k)}) \leq (k+1)N(r, w) + (2k-1)N_x(r, w) + O(1).$$

The above results can also be found in [28].

We say that the polynomial $P(z, w)$ in w is defined on $\Omega(f)$, if $a_i(z) \in \Omega(f)$ ($0 \leq i \leq m$) and the solution of $P(z, w) = 0$ is an exactly m -valued algebroid function, i.e., all its single-valued branches are mutually different.

P. C. Rosenbloom [38] was the first to utilize Nevanlinna theory to quantify the fix-points of composite functions. As an extension of Rosenbloom's result

[38], Prokopovich [36] studied the question of the number of fix-points of the composition of a rational function $R(z)$ with a meromorphic function $f(z)$; in the case of $f(z)$ is an entire function, he proved

$$(25) \quad \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/(R(f) - \gamma(z)))}{T(r, f)} \geq m - 1 + \delta(0, f),$$

where $\gamma(z)$ is a small meromorphic function with respect to $f(z)$. This confirms Rosenbloom's conjecture in [38]. Actually, Rosenbloom [38] had derived that the lim sup of the quantity on the left side of (25) is greater than or equal to 1 when $\gamma(z) \equiv z$. And (25) is a direct consequence of the following result due to Prokopovich [36].

Theorem 3.1 *Let $f(z)$ be a transcendental meromorphic function and the polynomial $P(z, w)$ in w be defined on $\Omega(f)$ with $m \geq 2$. Assume that among m single-valued branches of the algebroid function determined by $P(z, w)$, there are q functions linearly independent over the field of complex numbers. Then*

$$(26) \quad m - 1 - q(1 - \delta(\infty, f)) + \hat{\delta}(0, f) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/P(z, f))}{T(r, f)}$$

where

$$\hat{\delta}(0, f) = \begin{cases} \delta(0, f) & \text{if } P(z, 0) \not\equiv 0; \\ 0 & \text{if } P(z, 0) \equiv 0. \end{cases}$$

Based on Theorem 3.1, Prokopovich [36] conjectured the estimate

$$(27) \quad m - 2 + \delta(\infty, f) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/P(z, f))}{T(r, f)}.$$

It is easy to see from Theorem 3.1 that for a rational function $R(z)$ with degree ≥ 2 and a transcendental entire function $f(z)$, the composition $R(f)$ has infinitely many fix-points, as was dealt with in several papers, particularly, Gross–Yang [24] and Gross–Osgood [21]. Gross and Yang [24] proved this in the case when $R(z)$ is a polynomial of degree ≥ 3 and $f(z)$ is a transcendental meromorphic function, and characterized all the forms of $f(z)$ when $R(z)$ is of degree 2 and $R(f)$ has only finitely many fix-points. Furthermore, it was shown by Gross–Osgood [21] that the result still holds in the case when $R(z)$ is a rational function of degree ≥ 3 and $f(z)$ is a transcendental meromorphic function. Clearly, all the results we mentioned above can be derived from Prokopovich's conjecture.

In this section, by the methods of Steinmetz [40] which were used to extend the second fundamental theorem of Nevanlinna to the small meromorphic functions, we are able to confirm Prokopovich's conjecture and generalize some of his

other results in [36] to the case where $f(z)$ is a transcendental meromorphic function. More precisely, we shall prove the following, which contains Prokopovich's conjecture as a special case.

Theorem 3.2 *Let $f(z)$ be given as in Theorem 3.1 and the polynomial $P(z, w)$ in w be defined on $\Omega(f)$ with $m \geq 2$.*

Then

$$(28) \quad m - 2 + \delta(\infty, f) + \hat{\delta}(0, f) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/P(z, f))}{T(r, f)},$$

where $\hat{\delta}(0, f)$ is defined as in Theorem 3.1.

Proof Let $w(z)$ be the algebroid function determined by $P(z, w) = 0$. Through the branch points of $w(z)$, we draw a non-self-intersecting open polygon Γ_w to cut the complex plane such that $C \setminus \Gamma_w$ is a connected domain and $w(z)$ has exactly m single-valued analytic branches w_j ($1 \leq j \leq m$) on the domain.

For $s \geq 1$ and on $C \setminus \Gamma_w$, we span the C -linear space A_s by the functions $\prod_{j=1}^s w_j^{n_j}$ ($n_j \geq 0$, $\sum_{j=1}^s n_j = s$). As Steinmetz [40] did, we can deduce for the dimension d_s of A_s

$$\inf \frac{d_{s+1}}{d_s} \leq 1.$$

Let $b_1, \dots, b_p$ ($p = d_s$) be a basis of A_s and $B_1, \dots, B_q$ ($q = d_{s+1}$) be a basis of A_{s+1} . We consider the auxiliary function

$$W(f) := W(B_1, \dots, B_q, fb_1, \dots, fb_p),$$

where the right side is the Wronskian determinant. The function $W(f)$ is a single-valued branch of some finite-valued algebroid function, the other branches $W_i(f)$ of which we can obtain by continuing $W(f)$ along different paths intersecting Γ_w . Since $\{b_j; 1 \leq j \leq p\}$ and $\{B_j; 1 \leq j \leq q\}$ are the bases of, respectively, A_s and A_{s+1} , we can write

$$(29) \quad W_i(f) = c_i W(f),$$

where c_i is a non-zero complex number. Then according to [40], we have

$$(30) \quad W(f) = f^p \Psi \left(\frac{f'}{f} \right), \quad C \setminus \Gamma_w,$$

where $\Psi(f'/f)$ is a polynomial in f'/f , with the coefficients being rational functions of b_i 's and B_i 's and their derivatives, and

$$(31) \quad W(f - w_j) = W(f) \quad (1 \leq j \leq m).$$

By Lemma 3.1, we have

$$m(r, w^{(k)}) \leq m(r, w) + O(\log r T(r, w)), \quad r \notin E,$$

$$N(r, w^{(k)}) \leq (k+1)N(r, w) + 2(m-1)(2k-1)T(r, w) + O(1)$$

and hence

$$T(r, w^{(k)}) = O(T(r, w)) = O(\max\{T(r, a_j); 0 \leq j \leq m\}) = S(r, f).$$

It follows from properties of the Wronskian determinant [39] that

$$m(r, W(f)) \leq pm(r, f) + S(r, f)$$

and

$$N(r, W(f)) \leq (p+q)N(r, f) + S(r, f).$$

Put

$$\Psi_j = \Psi \left(\frac{(f - w_j)'}{f - w_j} \right)$$

and

$$A_j = [(w_j - w_1) \cdots (w_j - w_{j-1})(w_j - w_{j+1}) \cdots (w_j - w_m)]^{-1},$$

$1 \leq j \leq m$, on $C \setminus \Gamma_w$, $A_{m+1} = 1$, $w_{m+1} = 0$.

Assume that $P(z, 0) \not\equiv 0$. Now it follows from (30) and (31) that on $C \setminus \Gamma_w$

$$\begin{aligned} \log^+ \left| \frac{1}{P(z, f)} + \frac{1}{f} \right| &= \log^+ \left| \sum_{j=1}^{m+1} \frac{A_j}{f - w_j} \right| \\ &\leq \log^+ \sum_{j=1}^{m+1} \frac{|A_j|}{|f - w_j|} = \log^+ \sum_{j=1}^{m+1} \frac{|A_j| |\Psi_j|^{1/p}}{|W(f)|^{1/p}} \\ &\leq \frac{1}{p} \log^+ \frac{1}{|W(f)|} + \log^+ \sum_{j=1}^{m+1} |A_j| |\Psi_j|^{1/p}, \end{aligned}$$

and hence from (29) and Nevanlinna's first fundamental theorem for algebroid function that

$$\begin{aligned} m \left(r, \frac{1}{P(z, f)} + \frac{1}{f} \right) &\leq \frac{1}{p} m \left(r, \frac{1}{W(f)} \right) + S(r, f) \\ &\leq \frac{1}{p} T \left(r, \frac{1}{W(f)} \right) + S(r, f) \\ &\leq T(r, f) + \frac{q}{p} N(r, f) + S(r, f). \end{aligned}$$

Then a result of Mokhon'ko [34] leads to

$$\begin{aligned}
 (m+1)T(r, f) &\leq T\left(r, \frac{1}{P(z, f)} + \frac{1}{f}\right) + S(r, f) \\
 &\leq m\left(r, \frac{1}{P(z, f)} + \frac{1}{f}\right) + N\left(r, \frac{1}{P(z, f)}\right) + N\left(r, \frac{1}{f}\right) + S(r, f) \\
 &\leq T(r, f) + \frac{q}{p}N(r, f) + N\left(r, \frac{1}{P(z, f)}\right) + N\left(r, \frac{1}{f}\right) + S(r, f),
 \end{aligned}$$

and furthermore

$$m \leq \frac{q}{p}(1 - \delta(\infty, f)) + \liminf_{r \rightarrow \infty} \frac{N(r, 1/P(z, f))}{T(r, f)} + (1 - \delta(0, f)) \quad (r \notin E).$$

(28) follows from this.

In the case when $P(z, 0) \equiv 0$, by considering the function $1/P(z, f)$ instead of $1/P(z, f) + 1/f$, we follow the same steps as above and arrive at (28) for $\hat{\delta}(0, f) = 0$.

Theorem 3.2 follows. $\square$

The following is a direct consequence of Theorem 3.2, which is due to Prokopovich [36] when $f(z)$ is a transcendental entire function.

Corollary 3.3 *Let $f(z)$ be a transcendental meromorphic function and $R(z)$ be a rational function of degree $m \geq 1$ and $\gamma(z) \in \Omega(f) \setminus C$. Then*

$$(32) \quad m - 2 + \delta(\infty, f) + \delta(0, f) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/(R(f) - \gamma))}{T(r, f)}.$$

By virtue of Theorem 3.2, we also have the following, which was obtained in [36] when $f(z)$ is a transcendental entire function.

Corollary 3.4 *Let $f(z)$ be a transcendental meromorphic function and $w(z)$ an exactly m -valued algebroid function such that $T(r, w) = S(r, f)$. Then*

$$(33) \quad 1 - \frac{2}{m} + \frac{1}{m}\delta(\infty, f) + \frac{1}{m}\hat{\delta}(0, f) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N^*(r, 1/(f - w))}{T(r, f)},$$

where $\hat{\delta}(0, f) = 0$, if w contains a branch which identically vanishes, or else $\hat{\delta}(0, f) = \delta(0, f)$.

Proof Assume that $w(z)$ is defined by the equation

$$P(z, w) = 0.$$

From a result of Valiron [44] (cf. [28, p.81]), we can assume that the coefficients $a_j(z)$'s of polynomial $P(z, w)$ in w are elements of $\Omega(f)$. If in some neighborhood of $z = z_0$, $f(z) = w(z) + (z - z_0)^p h(z)$, $h(z_0) \neq 0$, ∞ , $p > 0$, then by simple calculation, we have

$$\begin{aligned} P(z, f) &= a_m(z)(w(z) + (z - z_0)^p h(z))^m + \cdots \\ &\quad + a_1(z)(w(z) + (z - z_0)^p h(z)) + a_0(z) \\ &= (z - z_0)^p h(z) A(z), \end{aligned}$$

where $A(z)$ is such that

$$A(z_0) = \left. \frac{\partial P(z, w)}{\partial w} \right|_{\substack{z=z_0 \\ w=w(z_0)}}.$$

It is evident that if $A(z_0) = 0$, then z_0 is a branch-point of $w(z)$ and furthermore

$$N^* \left(r, \frac{1}{f - w} \right) \geq \frac{1}{m} N \left(r, \frac{1}{P(z, f)} \right) - N_x(r, w).$$

Then (33) follows from Lemma 3.1 and a combination of this with Theorem 3.2. $\square$

Furthermore, from Corollary 3.4 and by the similar methods used in the proof of [36, Theorem 4], we can obtain a refined result on the number of fix-points of the composition $f(P(z))$, where $f(z)$ is a transcendental meromorphic function and $P(z)$ is a polynomial of degree ≥ 2 . Actually, we shall derive

Theorem 3.5 *Let $f(z)$ be given as in Theorem 3.2 and $P(z)$ be a polynomial of degree $m \geq 2$. Assume that $F(z) = f(P(z))$. Then the inequality*

$$(34) \quad 1 - \frac{2}{m} + \frac{1}{m} \delta(\infty, F) + \frac{1}{m} \delta(0, F) \leq \liminf_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N(r, 1/(F - z))}{T(r - \xi, F)}$$

holds, where ξ is a positive number depending only on $P(z)$.

Proof Write $P(z) = c_m z^m + \cdots + c_0$, $c_i \in \mathbb{C}$, $c_m \neq 0$. And let $w(z)$ be an algebraic function determined by the equation

$$(35) \quad P(z) = P(zw).$$

It is evident that $zw(z) \rightarrow \infty$, as $z \rightarrow \infty$, for the left side of (35) tends to infinity. Then we can write

$$P(z) = c_m z^m (1 + O(1/z)), \quad P(zw(z)) = c_m (zw(z))^m (1 + O(1/zw(z)))$$

as $z \rightarrow \infty$. It follows from (35) that $|z| \sim |zw(z)|$ and

$$\begin{aligned} w(z)^m &= \frac{1 + O(1/z)}{1 + O(1/zw)} \\ &= 1 + O(1/z), \end{aligned}$$

as $z \rightarrow \infty$. This implies that $zw(z)$ has exactly m single-valued branches and the existence of a positive number ξ such that for all sufficiently large r

$$|w(z)| < 1 + \frac{\xi}{r}.$$

Then

$$n\left(r + \xi, \frac{1}{f(P) - z}\right) \geq \frac{1}{m}n\left(r, \frac{1}{f(P) - zw}\right),$$

and furthermore

$$N\left(r + \xi, \frac{1}{f(P) - z}\right) \geq N^*\left(r, \frac{1}{f(P) - zw}\right).$$

Since $zw(z)$ is an exactly m -valued algebraic function, by Corollary 3.4, we immediately derive (34), and Theorem 3.5 follows. $\square$

Finally, we would like to conclude this section with the following conjecture:

The quantity ξ in (34) can be discarded.

4. Composite entire functions

Throughout this section, let $f(z)$ and $g(z)$ be two transcendental entire functions and $P(z)$ be a non-constant polynomial. Then we can write

$$(36) \quad f(g(z)) = P(z) + \alpha(z)e^{\beta(z)},$$

where $\beta(z)$ is an entire function and $\alpha(z)$ is a Weierstraß product for zeros of $f(g(z)) - P(z)$.

What we shall do here is to establish some quantitative estimates of the number of the zeros of $f(g(z)) - P(z)$ in terms of either the maximum modulus of $f(g)$ or Nevanlinna characteristic of $g(z)$. The arguments that we shall use here are mainly the Wiman–Valiron type of theorems which were employed by Bergweiler [5] to prove the Gross conjecture mentioned in Section 1; Steinmetz's theorem [41], which is a very powerful tool in studying factorization theory of meromorphic functions and which enables one to reduce a fairly complicated equation involving

composite functions into a much simpler form, and the estimate of the maximum modulus of the canonical product in terms of its zeros listed in Section 2.

When $0 < \lambda(f) \leq \rho(f) < \infty$, Bergweiler [6] compared the growth of the k -th iterate of $f(z)$ with the number of its fix-points of k -order and proved the following:

$$(37) \quad \limsup_{r \rightarrow \infty} \frac{\log_k N(r, 1/(f_k - z))}{\log r} = \rho_k(f_k) < \infty,$$

that is, the two of them are equivalent in the sense of k -order of entire function. (37) was shown already by Zheng–Yang [48] when either $0 < \lambda(f)$ or $T(r, f) \leq O((\log r)^p)$, $p > 1$.

In terms of the maximum modulus and characteristic function, Zheng–Yang [47] was able to derive the following estimate:

There exists a set I of infinite logarithmic measure such that for $r \in I$,

$$(38) \quad N\left(r, \frac{1}{f(g) - P}\right) > \min\{dT(r^{1/3}, g), [\log M(r^{1/3}, f(g))]^{1/\log r}\},$$

and if $\lambda(f) > 0$,

$$(39) \quad N\left(r, \frac{1}{f(g) - P}\right) > [\log M(r^{1/3}, f(g))]^{1/4}.$$

The factors “ $1/3$ ”, “ $1/\log r$ ” and “ $1/4$ ” above result mainly from the estimate of the growth of canonical product by the number of its zeros. Our main purposes of this section are to improve (38) without any assumptions on the growths of $f(z)$ and $g(z)$ (which is Theorem 4.2) and (39) under the assumption that $\lambda(f) > 0$ and $(\log r)^3 = O(v(r, g))$ (Theorem 4.4). Finally, we ought to mention here the work of Langley [33], in which a better estimate was established in the case when $\rho(f(g)) < \infty$. Actually, it was proved that if $\rho(f(g)) < \infty$, then $\delta(0, f(g) - Q) < 1$, i.e.,

$$N\left(r, \frac{1}{f(g) - Q}\right) \neq o(T(r, f(g))),$$

where Q is a non-constant rational function. This confirms Conjecture 2 in the special case.

Below we list some lemmas which will be used to prove our results. First of all, we need a different form of Nevanlinna’s second fundamental theorem, which can be found in [46, p.68].

Lemma 4.1 *Let $h(z)$ be meromorphic in $|z| < R (\leq \infty)$. If $h(0) \neq 0, 1, \infty$ and $h'(0) \neq 0$, then for $0 < r < R$,*

$$T(r, h) < 2 \left\{ N(R, h) + N\left(R, \frac{1}{h}\right) + N\left(R, \frac{1}{h-1}\right) \right\} + 191 \\ + 4 \log^+ |h(0)| + 2 \log^+ \frac{1}{R|h'(0)|} + 12 \log^+ \frac{r}{R-r}.$$

Applying Lemma 4.1, we obtain the following lemma, which will be used in the proof of our main theorem.

Lemma 4.2 *Let $h(z)$ be holomorphic in $|z| \leq R$. If $h(0) \neq 0, 1$, then for $0 < r < R$, we have*

$$\log M(r, h) < \frac{3r+R}{R-r} \left(2N(R) + 195 + 4 \log^+ |h(0)| + 12 \log^+ \frac{2R}{R-r} \right),$$

where

$$N(R) := N\left(R, \frac{1}{h}\right) + N\left(R, \frac{1}{h-1}\right).$$

Proof First of all, assume $R = 1$. If $|h'(0)| \geq 1$, then by Lemma 4.1, we immediately conclude that

$$\log M(r, h) \leq \frac{\frac{1+r}{2} + r}{\frac{1+r}{2} - r} T\left(\frac{1+r}{2}, h\right) \\ < \frac{3r+1}{1-r} \left(2N(1) + 191 + 4 \log^+ |h'(0)| + 12 \log^+ \frac{2}{1-r} \right).$$

Now we assume $|h'(0)| < 1$. We take a point t on $|z| = r$ such that $|h(t)| = M(r, h)$. Draw a segment $\overline{0t}$ connecting 0 and t . Below we need to treat two cases.

Case (a). The inequality $|h'(z)| < 1$ always holds on the segment $\overline{0t}$. Obviously,

$$|h(t)| \leq |h(0)| + \left| \int_{\overline{0t}} h'(z) dz \right|,$$

and hence

$$\log M(r, h) \leq \log^+ |h(0)| + \log 2.$$

Case (b). There exists a point z_0 on the segment $\overline{0t}$ such that $|h'(z_0)| = 1$ and for $z \in \overline{0z_0}$, $|h'(z)| < 1$, and further

$$|h(z_0)| \leq |h(0)| + 1.$$

Set $d := |z_0 - t| < r$. Then $h(z)$ is still holomorphic in $|z - z_0| \leq d + 1 - r$. Applying Lemma 4.1 to the previous disk implies that

$$\begin{aligned} \log M(r, h) &= \log M(d, z_0, h) \\ &\leq \frac{d + \frac{1-r}{2} + d}{d + \frac{1-r}{2} - d} T \left(d + \frac{1-r}{2}, z_0, h \right) \\ &\leq \frac{3r+1}{1-r} \left(2N(1) + 191 + 4 \log^+ |h(z_0)| + 12 \log^+ \frac{d + \frac{1-r}{2}}{d + 1 - r - (d + \frac{1-r}{2})} \right) \\ &\leq \frac{3r+1}{1-r} \left(2N(1) + 195 + 4 \log^+ |h(0)| + 12 \log^+ \frac{2}{1-r} \right). \end{aligned}$$

When $R \neq 1$, set $f(z) = h(Rz)$. Certainly, $f(z)$ is holomorphic in $|z| < 1$. Then from the above discussion, we have

$$\begin{aligned} \log M(r, h) &= \log M\left(\frac{r}{R}, f\right) \\ &\leq \frac{3\frac{r}{R} + 1}{1 - \frac{r}{R}} \left(2N(R) + 195 + 4 \log^+ |h(0)| + 12 \log^+ \frac{2}{1 - \frac{r}{R}} \right), \end{aligned}$$

which easily produces the desired inequality of Lemma 4.2.

We denote by $v(r, g)$ and $\mu(r, g)$, respectively, the central index and the maximum term of an entire function $g(z)$.

Lemma 4.3 *Let $g(z)$ be a transcendental entire function. Assume that $c > 0$, $\varepsilon > 0$, $\eta > 0$ and $\gamma > 1/2$. Suppose that $|z_0| = r$, $|g(z_0)| \geq \eta M(r, g)$ and $|\tau| < cv(r, g)^{-\gamma}$. Then*

$$(40) \quad g(z_0 e^\tau) \sim g(z_0) e^{v(r, g)\tau} \quad (r \notin F);$$

$$(41) \quad g'(z_0 e^\tau) \sim \frac{v(r, g)}{z_0 e^\tau} g(z_0) e^{v(r, g)\tau} \quad (r \notin F);$$

$$(42) \quad \log M(r, g) \sim \log M(r, g') \quad (r \notin F);$$

$$(43) \quad v(r, g) \leq (\log \mu(r, g))^{1+\varepsilon} \leq (\log M(r, g))^{1+\varepsilon} \quad (r \notin F);$$

$$(44) \quad \log M(r, g) \leq (1 + o(1))(\log \mu(r, g)) \leq (1 + o(1))\nu(r, g) \log r \quad (r \notin F).$$

Lemma 4.4 *Let $g(z)$ be given as in Lemma 4.3 and assume that $c > 0$, $\eta > 0$ and $1 \geq \gamma > 1/2$. Suppose that $|z_o| = r \notin F$, $|g(z_o)| \geq \eta M(r, g)$. Then there exists a function $\tau(z)$ defined and analytic on $|z - z_o| \leq cr\nu(r, g)^{-\gamma}$ satisfying*

$$|\tau(z)\nu(r, g) - 2\pi i| = o(1),$$

$$g(ze^{\tau(z)}) = g(z).$$

The two lemmas above are in essence contained in Wiman–Valiron theory [43] and the latter was explicitly developed by Bergweiler [5] and Zheng–Yang [47].

Now let us recall two standard measures of a set on the positive real axis, which we often meet in the sequel. Let H be a Lebesgue-measurable set on part of the real axis greater than 1. We define the linear measure and logarithmic measure of H , mH and $\text{lm}H$, by, respectively,

$$mH = \int_H dt \quad \text{and} \quad \text{lm}H = \int_H \frac{1}{t} dt,$$

and the upper and lower densities of H by

$$\overline{\text{dens}}H = \lim_{r \rightarrow \infty} \sup \inf \frac{1}{r} \int_{H \cap [1, r]} dt,$$

and the upper and lower logarithmic densities of H by

$$\overline{\log \text{dens}}H = \lim_{r \rightarrow \infty} \sup \inf \frac{1}{\log r} \int_{H \cap [1, r]} \frac{dt}{t}.$$

Obviously $0 \leq \text{lm}H \leq mH$. Therefore if it has finite linear measure, H is of finite logarithmic measure. And it is also well known and easily proved that

$$0 \leq \underline{\text{dens}}H \leq \underline{\log \text{dens}}H \leq \overline{\log \text{dens}}H \leq \overline{\text{dens}}H \leq 1,$$

and if $\text{lm}H < \infty$, then $\overline{\text{dens}}H = 0$ (cf. Barry [4]).

In order to derive Lemma 4.7, we first state two known results due to, in turn, Hayman [26] and Bergweiler [7].

Lemma 4.5 *Let $g(z)$ be a transcendental meromorphic function of finite lower order. Then for arbitrary two real numbers $A > 1$ and $B > 1$, the set*

$$J = \{r; T(Ar, g) \geq BT(r, g)\}$$

has a lower logarithmic density

$$(45) \quad \underline{\log \text{dens}} J \leq \lambda(g) \frac{\log A}{\log B}.$$

Lemma 4.6 *Let $s(x)$ and $k(x)$ be non-negative, non-decreasing and convex for $x \geq 0$. Assume that $s(x) \leq k(x)$ for all $x \geq 0$. Then for arbitrary $D > 1$, we have $s'(x) \leq Dk'(x)$ on a set I satisfying*

$$(D - 1)/D \leq \underline{\text{dens}} I \leq \underline{\log \text{dens}} I.$$

Now we can derive the following

Lemma 4.7 *Let $g(z)$ and $\beta(z)$ be two transcendental entire functions. If there exists a set G having positive lower logarithmic density such that for a positive number a ,*

$$(46) \quad aT(r, g) \leq T(r, \beta), \quad r \in G,$$

then for arbitrary $\varepsilon > 0$, there exists a subset H of G with a positive lower logarithmic density such that

$$(47) \quad v(r, g) \leq v(r, \beta)^{1+\varepsilon}, \quad r \in H.$$

Proof For $\varepsilon > 0$, we can choose a positive number ε' such that

$$1 + \varepsilon = \frac{(1 + \varepsilon')^2}{1 - \varepsilon'}.$$

From a classical lemma of Borel [27, Lemma 2.4], we have

$$\log M(r, g) < T(r, g)^{1+\varepsilon'/2}, \quad r \notin F,$$

and from (43), (46) and (44),

$$\begin{aligned} v(r, g) &\leq (\log M(r, g))^{1+\varepsilon'/2} \leq T(r, g)^{(1+\varepsilon'/2)^2} \\ &\leq \left(\frac{1}{a} \log M(r, \beta) \right)^{(1+\varepsilon'/2)^2} \\ &\leq (v(r, \beta) \log r)^{(1+\varepsilon')^2}, \quad r \in G \setminus F. \end{aligned}$$

If

$$(\log r)^{(1+\varepsilon')^2} \leq v(r, g)^{\varepsilon'}, \quad r \in G \setminus F,$$

then we immediately deduce (47). Otherwise, the above inequality does not hold inside a set of r with infinite logarithmic measure so that $g(z)$ has zero lower order. Then it follows from Lemma 4.5 that

$$T(2r, g) < 2T(r, g), \quad r \notin J,$$

where J is a set of r with zero lower logarithmic density and, furthermore,

$$\log M(r, g) \leq 3T(2r, g) < 6T(r, g) < \frac{6}{a} \log M(r, \beta), \quad r \in G \setminus J.$$

Set

$$x = \log r, \quad s(x) = \max\{0, \log \mu(r, g)\}$$

and

$$k(x) = \max \left\{ s(x), \frac{7}{a} \log \mu(r, \beta) \right\}.$$

Then Lemma 4.7 follows from Lemma 4.6 and the fact that

$$\frac{d \log \mu(r, h)}{d \log r} = v(r, h)$$

for an entire function h , except for the discontinuities of $v(r, h)$.

The following is due to Clunie [13].

Lemma 4.8 *Let $f(z)$ and $g(z)$ be two transcendental entire functions. Then*

$$M(r, f(g)) = M((1 - o(1))M(r, g), f), \quad r \notin F.$$

Finally, we recall a modified version of Steinmetz's Theorem [22].

Lemma 4.9 *Let $F_0, F_1, \dots, F_m$ be not identically vanishing entire functions and let $h_0, h_1, \dots, h_m$ be arbitrary meromorphic functions ($m \geq 1$). Assume that $g(z)$ is a non-constant entire function, K a positive number and $\{r_j\}$ an unbounded monotone increasing sequence of positive real numbers such that, for each j ,*

$$T(r_j, h_i) \leq KT(r_j, g) \quad (i = 0, 1, \dots, m),$$

and

$$T(r_j, g') \leq (1 + o(1))T(r_j, g).$$

If F_i and h_i ($i = 0, 1, \dots, m$) satisfy

$$F_0(g)h_0 + F_1(g)h_1 + \dots + F_m(g)h_m = 0,$$

then there exist polynomials in z , $P_0, P_1, \dots, P_m$, not all zero, such that

$$P_0F_0 + P_1F_1 + \dots + P_mF_m \equiv 0.$$

We begin with

Theorem 4.1 *Let $f(z)$, $g(z)$, $P(z)$, $\alpha(z)$ and $\beta(z)$ be given as in the beginning of the section and such that (36) holds. Assume that for a set I on the positive real axis with infinite logarithmic measure, we have*

(i) for $d > 1$

$$(48) \quad d \log M(r + \eta, \alpha) \leq \log M(r, f(g)), \quad r \in I$$

and

(ii) for $1 > \varepsilon \geq 0$

$$(49) \quad v(r, g) < v(r, \beta)^{1+\varepsilon}, \quad r \in I.$$

Then

$$(50) \quad K \log M(r - 7\eta, f(g)) < N \left(r, \frac{1}{f(g) - P} \right), \quad r \in I \setminus F,$$

where

$$K = \frac{1}{6} \left[1 - \frac{25}{2d} \right], \quad \eta = crv(r, g)^{-\gamma},$$

$$\gamma = \frac{1}{1+\varepsilon}, \quad c = \arctan \frac{16}{13} + \frac{\pi}{2}.$$

Proof For $r \in I$, we choose a point z_0 with $|z_0| = r$ such that

$$(51) \quad |f(g(z_0))| = M(r, f(g)) = M((1 - o(1))M(r, g), f) \quad (r \in I \setminus F).$$

The latter comes from Lemma 4.8. Then it follows that

$$|g(z_0)| \geq (1 - o(1))M(r, g),$$

and from (36), (48) and (51) that

$$(52) \quad |\beta(z_o)| > \operatorname{Re} \beta(z_o) = \log \left| \frac{f(g(z_o)) - P(z_o)}{\alpha(z_o)} \right|$$

$$(53) \quad \geq \log M(r, f(g)) - \log M(r, \alpha) - O(\log r)$$

$$(54) \quad \geq \frac{d}{d+2} \log M(r, e^\beta)$$

$$(55) \quad \geq \frac{d(1-o(1))}{d+2} M(r, \beta) \quad (r \in I \setminus F).$$

By applying Lemma 4.4 to $g(z)$ at z_o , we deduce the existence of an analytic function $\tau(z)$ on $|z - z_o| \leq 20crv(r, g)^{-\gamma}$, $c := \arctan((d+2)/d) + \pi/2$ and $1/2 < \gamma = 1/(1+\varepsilon) \leq 1$, such that

$$|\tau(z)v(r, g) - 2\pi i| = o(1) \quad \text{and} \quad g(ze^{\tau(z)}) = g(z).$$

Set $k(z) = ze^{\tau(z)}$ and

$$(56) \quad h(z) = \frac{f(g(z)) - P(z)}{P(k(z)) - P(z)}.$$

A simple calculation shows that $h(z)$ is analytic in $|z - z_o| \leq 12cv(r, g)^{-\gamma}$.

Define $z_1 = z_o e^{i\theta}$ where $\theta \in \mathbb{R}$ (real set) is chosen such that $0 \leq \operatorname{Re} \beta(z_1) < 1$, $h(z_1) \neq 0, 1$ and

$$\begin{aligned} |v(r, \beta)\theta - \pi/2| &= |\arg \beta(z_o) + o(1)| \\ &= |\arctan(\operatorname{Im} \beta(z_o)/\operatorname{Re} \beta(z_o)) + o(1)| \\ &< \arctan((d+2)/d). \end{aligned}$$

We can do this by (40) in Lemma 4.3 and (55). Then it is easy to see that

$$|z_1 - z_o| = r|e^{i\theta} - 1| \sim r|\theta| < crv(r, \beta)^{-1},$$

and from (49) that

$$|z_1 - z_o| < crv(r, g)^{-\gamma} = \xi \quad (\text{say}),$$

where $\gamma = 1/(1 + \varepsilon)$. This implies that $h(z)$ is well defined and is analytic in $|z - z_1| < 10\xi$.

We now estimate the following quantity:

$$(57) \quad N(\xi) := N(3\xi, z_1, h) + N\left(3\xi, z_1, \frac{1}{h}\right) + N\left(3\xi, z_1, \frac{1}{h-1}\right).$$

To this end, in the disk $|z - z_1| < 3\xi$ we observe

$$|z| < |z_1| + 3\xi < |z_0| + 4\xi = r + 4\xi;$$

$$\begin{aligned} |k(z)| &= |ze^{\tau(z)}| < (r + 4\xi)e^{|\tau(z)|} \\ &< (r + 4\xi)(1 + 2cv(r, g)^{-1}) < r + 7\xi \quad (r \in I \setminus F). \end{aligned}$$

Set $\chi = r + 7\xi$. It follows from $g(k(z)) = g(z)$, (56) and (57) that

$$(58) \quad N(\xi) < N\left(\chi, \frac{1}{f(g) - P}\right).$$

Now we need to estimate $\log^+ |h(z_1)|$ as follows:

$$\begin{aligned} (59) \quad \log^+ |h(z_1)| &= \log^+ \left| \frac{\alpha(z_1)e^{\beta(z_1)}}{P(k(z_1)) - P(z_1)} \right| \\ &< \log M(r, \alpha) + O(\log rv(r, g)) \\ &< \frac{1}{d} \log M(r, f(g)) + O(\log rv(r, g)). \end{aligned}$$

Thus noting $h(z_1) \neq 0, 1$ and by applying Lemma 4.2 to $h(z)$ in $|z - z_1| < 3\xi$ and from (58) and (59), we obtain

$$\begin{aligned} (60) \quad \log M(\xi, z_1, h) &< \frac{3\xi + 3\xi}{3\xi - \xi} \left(2N\left(\chi, \frac{1}{f(g) - P}\right) + 195 + 4\log^+ |h(z_1)| \right. \\ &\quad \left. + 12\log^+ \frac{6\xi}{3\xi - \xi} \right) \\ &< 6N\left(\chi, \frac{1}{f(g) - P}\right) + \frac{12}{d} \log M(r, f(g)) \\ &\quad + O(\log rv(r, g)), \quad r \in I \setminus F. \end{aligned}$$

On the other hand, we have from the definition of $h(z)$ that

$$\log |h(z_0)| > \log M(r, f(g)) - O(\log rv(r, g)), \quad r \in I \setminus F.$$

By combining (60) with the above inequality and noting

$$\log |h(z_0)| < \log M(\xi, z_1, h),$$

we easily conclude the desired inequality (50). Thus Theorem 4.1 follows. $\square$

With the help of Theorem 4.1, Steinmetz's result (Lemma 4.9) and Theorem 2.1 concerning the estimate of canonical products, we are able to improve the estimate in (38). First, however, we need the following key lemma.

Lemma 4.10 *Let $f(z)$ and $g(z)$ be two transcendental entire functions and $P(z)$ be a non-constant polynomial. Assume that there exist a set I on the positive real axis with infinite logarithmic measure and a positive number a such that*

$$(61) \quad N\left(r, \frac{1}{f(g) - P}\right) \leq aT(r, g) \quad (r \in I).$$

Then we can construct two entire functions $\alpha(z)$ and $\beta(z)$ such that (36) holds and satisfy

$$(62) \quad \log \log M(r, \alpha) = o\left(N\left(r, \frac{1}{f(g) - P}\right)\right) \quad (r \notin F),$$

and

$$(63) \quad T(r, g) < T(r, \beta) \quad (r \in I \setminus F).$$

Proof According to Theorem 2.1, we can construct two entire functions $\alpha(z)$ and $\beta(z)$ such that (36) and (62) hold. Thus we need only prove (63). From the lemma of logarithmic derivative (cf. [27]), (61) and (62) we have

$$m(r, \alpha'/\alpha) = O(\log r T(r, \alpha)) = o(N(r, 1/\alpha)) = o(T(r, g)) \quad (r \notin F),$$

so that

$$(64) \quad T(r, \alpha'/\alpha) < 2aT(r, g) \quad (r \notin F).$$

Suppose that $J = \{r; T(r, g) \geq T(r, \beta)\}$ has an infinite logarithmic measure. Thus we have

$$(65) \quad T(r, \beta') \leq 2T(r, \beta) < 2T(r, g) \quad (r \in J \setminus E).$$

By differentiating both sides of (36), we have the following equation

$$(66) \quad f'(g)g' = \left(\frac{\alpha'}{\alpha} + \beta' \right) (f(g) - P) + P'.$$

Then according to (64), (65) and Lemma 4.9, there exist three polynomials $P_1(z)$, $P_2(z)$ and $P_3(z)$, not all zero, such that

$$P_1 f' + P_2 f + P_3 = 0,$$

and thus

$$(67) \quad P_1(g)f'(g) + P_2(g)f(g) + P_3(g) = 0.$$

Eliminating $f'(g)$ from (66) and (67) gives

$$f(g) = -\frac{g'P_3(g) + P'P_1(g)}{P_1(g)(\alpha'/\alpha + \beta')} + P.$$

It is easy to see that $T(r, f(g)) = O(T(r, g))$ ($r \in J \setminus F$), which is a contradiction, since $T(r, g) = S(r, f(g))$ (cf. [20]). Thus we show (63), possibly outside a set F of finite logarithmic measure.

Theorem 4.2 *Let $f(z)$ and $g(z)$ be two transcendental entire functions and $P(z)$ be a non-constant polynomial. Then there exists a set I having the lower logarithmic density one such that*

$$(68) \quad \lim_{\substack{r \rightarrow \infty \\ r \in I}} \frac{N(r + \eta, 1/(f(g) - P))}{T(r, g)} = \infty,$$

where $\eta = 7c \nu(r, g)^{-\gamma}$, $\frac{1}{2} < \gamma < 1$ and $c = \arctan(16/13) + \pi/2$.

Proof Suppose conversely that Theorem 4.2 does not hold. Then there exists a set G with a positive lower logarithmic density such that

$$(69) \quad N\left(r + \eta, \frac{1}{f(g) - P}\right) \leq BT(r, g), \quad r \in G,$$

where B is a positive number. By Lemma 4.10, there exist two entire functions $\alpha(z)$ and $\beta(z)$ satisfying

$$(36) \quad f(g(z)) = P(z) + \alpha(z)e^{\beta(z)}$$

and for $\tilde{\eta} = \eta/7$,

$$(70) \quad \log \log M(r + \tilde{\eta}, \alpha) = o(N(r + \tilde{\eta}, 1/\alpha)), \quad r \notin F$$

and

$$T(r, g) < T(r, \beta) \quad (r \in G \setminus F).$$

Then by Lemma 4.7 there exists a subset H of G with positive lower logarithmic density such that

$$v(r, g)^\gamma < v(r, \beta), \quad \frac{1}{2} < \gamma < 1, \quad r \in H.$$

Put

$$J = \{r \in H; 13 \log M(r + \tilde{\eta}, \alpha) < \log M(r, f(g))\}.$$

If J has an infinite logarithmic measure, then from Theorem 4.1 we deduce that

$$\frac{1}{156} \log M(r, f(g)) < N\left(r + \eta, \frac{1}{f(g) - P}\right), \quad r \in J \setminus F.$$

Since $J \setminus F \subset H \setminus F \subset G$, we arrive at a contradiction to (69). Therefore, the set J has only finite logarithmic measure, that is, the inequality

$$(71) \quad 13 \log M(r + \tilde{\eta}, \alpha) \geq \log M(r, f(g)), \quad r \in H \setminus F$$

holds. Define the set S by

$$S := \{r \in H; \log \log M(r, f(g)) \leq T(r, g)\}.$$

Suppose that S has infinite linear measure. Set

$$R(z) = f(g(z)) - P(z).$$

By the lemma of logarithmic derivative

$$\begin{aligned} m(r, \frac{R'}{R}) &= O(\log r T(r, R)) \\ &= O(\log \log M(r, f(g)) + \log r) \\ &= O(T(r, g)), \quad r \in S \setminus E; \end{aligned}$$

furthermore by (69) and noting $S \subset G$, we have

$$T(r, R'/R) \leq N(r, 1/R) + O(T(r, g)) = O(T(r, g)), \quad r \in S \setminus E.$$

Now we can apply Lemma 4.9 to the equation

$$f'(g)g' = \frac{R'}{R}(f(g) - P) + P',$$

and by an argument similar to the proof of Lemma 4.10 obtain a contradiction. Thus S has finite linear measure, i.e., the inequality

$$\log \log M(r, f(g)) \geq T(r, g), \quad r \in H \setminus E$$

holds. By combining (71), (70) with the above inequality, we deduce that

$$T(r, g) \leq o\left(N\left(r + \eta, \frac{1}{f(g) - P}\right)\right), \quad r \in H \setminus F,$$

which is a contradiction; thus the proof of Theorem 4.2 is completed. $\square$

The following is an immediate consequence of Theorem 4.2 and Lemma 4.5.

Corollary 4.3 *Let $f(z)$, $g(z)$ and $P(z)$ be defined as in Theorem 4.2. Assume that $\lambda(g) < \infty$. Then there exists a set I having the lower logarithmic density one such that*

$$\lim_{\substack{r \rightarrow \infty \\ r \in I}} \frac{N(r, 1/(f(g) - P))}{T(r, g)} = \infty.$$

The following is an improvement of (39).

Theorem 4.4 *Let $f(z)$, $g(z)$ and $P(z)$ be given as in Theorem 4.2 and μ denote the quantity in (1) for the zeros of $f(g(z)) - P(z)$. Assume that*

$$(72) \quad \lambda(f) > 0 \quad \text{and} \quad (\log r)^3 = O(v(r, g)).$$

Then $\mu \geq 1$. Furthermore, for arbitrary $k > 1$, there exists a set I of infinite logarithmic measure such that if $\mu > 1$, then

$$(73) \quad K_1 \log M(r, f(g)) < N\left(kr, \frac{1}{f(g) - P}\right), \quad r \in I,$$

where $K_1 = \min\{1/156, (39K)^{-1} \log k\}$ and K is defined as in Theorem 2.2, and if $\mu = 1$, then

$$(74) \quad K_2 \log M(r, f(g)) < N\left(kr, \frac{1}{f(g) - P}\right) \log N\left(kr, \frac{1}{f(g) - P}\right), \quad r \in I,$$

where $K_2 = 2[39(3e - 3)]^{-1} \log k$.

Proof First we can always express $f(g(z))$ as

$$(75) \quad f(g(z)) = P(z) + \alpha(z)e^{\beta(z)},$$

where $\alpha(z)$ is a Weierstraß product for the zeros of $f(g(z)) - P(z)$ and $\beta(z)$ a suitable entire function.

Assume that we can construct the Weierstraß product $\alpha(z)$ such that for a subset U of the positive real set with infinite logarithmic measure and $k = \delta^3$, the inequality

$$(76) \quad 13 \log M(\delta r, \alpha) < \log M(r, f(g)), \quad r \in U$$

holds. Then from (75) and (76), we have the following estimate:

$$\begin{aligned} \log M(r, f(g)) &\leq \log M(r, \alpha) + \log M(r, e^\beta) + O(\log r) \\ &\leq \frac{1}{2} \log M(r, f(g)) + M(r, \beta), \quad r \in U. \end{aligned}$$

Thus it follows from the assumption $\lambda(f) > 0$ and Lemma 4.8 that for some positive number b

$$(77) \quad b \log M(r, g) \leq \log \left[\frac{1}{2} \log M(r, f(g)) \right] \leq \log M(r, \beta), \quad r \in U \setminus F,$$

and furthermore from (43) and (44) that

$$\begin{aligned} v(r, g) &< (\log M(r, g))^{1+\varepsilon} < \left(\frac{1}{b} \log M(r, \beta) \right)^{1+\varepsilon} \\ &\leq \left(\frac{2}{b} v(r, \beta) \log r \right)^{1+\varepsilon}, \quad r \in U \setminus F. \end{aligned}$$

The assumption that $(\log r)^3 \leq dv(r, g)$ for a positive number d implies

$$v(r, g)^\gamma < v(r, \beta), \quad r \in U \setminus F,$$

where $2/3 \leq \gamma < 1$. By applying Theorem 4.1, we immediately obtain

$$\frac{1}{156} \log M(r - 7\eta, f(g)) < N \left(r, \frac{1}{f(g) - P} \right), \quad r \in U \setminus F,$$

which contains (73) and (74), since $\eta/r \rightarrow 0$ and further $\mu \geq 1$.

Now we can assume that any Weierstraß product $\alpha(z)$ satisfying (36) is such that

$$(78) \quad 13 \log M(\delta r, \alpha) \geq \log M(r, f(g)), \quad r \notin F.$$

Then

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, 1/(f(g) - P))}{\log r} = \infty,$$

and further $\mu \geq 1$. Actually, otherwise, the order of $\alpha(z)$ would be finite, and so would be $f(g)$, which is impossible, since $\lambda(f) > 0$.

When $\mu > 1$, we can construct $\alpha(z)$, as we have done in (I) of Theorem 2.2, with the property that there exists an unbounded sequence $\{R_m\}$ such that

$$\log M(R_m, \alpha) < Kn \left(R_m, \frac{1}{f(g) - P} \right),$$

where K is defined as in Theorem 2.2. Set

$$V = \bigcup_{m=1}^{\infty} \{r; (1/\delta)R_m \leq r \leq R_m\}.$$

It is easy to see that V has infinite logarithmic measure. Then for $(1/\delta)R_m \leq r \leq R_m$ and $r \notin F$, from (78), we have

$$\begin{aligned} \log M \left(\frac{1}{\delta}r, f(g) \right) &\leq 13 \log M(r, \alpha) \leq 13 \log M(R_m, \alpha) \\ &< 13Kn \left(R_m, \frac{1}{f(g) - P} \right) < 13Kn \left(\delta r, \frac{1}{f(g) - P} \right) \\ &\leq \frac{13K}{\log \delta} N \left(\delta^2 r, \frac{1}{f(g) - P} \right), \end{aligned}$$

which leads to (73). For the case $\mu = 1$, by an argument similar to the above and (II) in Theorem 2.2, (74) will be easily derived.

Thus we complete the proof of Theorem 4.4. $\square$

Remark By analyzing the proof of Theorem 4.4, we still have the estimates (73) and (74), provided that the following condition, instead of (72), holds:

$$\log \log M(r, f(g)) \geq [\log M(r, g)]^\mu, \quad r \notin F,$$

and

$$(79) \quad v(r, g) \geq (\log r)^x,$$

where $1/2 < \mu \leq 1$, $2/(2\mu - 1) < x$.

Actually, it is obvious like (77) that the inequality

$$(80) \quad [\log M(r, g)]^\mu \leq \log M(r, \beta)$$

is valid. Choose a positive number ε so small that

$$\frac{2(1+\varepsilon)}{2\mu-1-\varepsilon} < x \quad \text{and} \quad y = \frac{x(1+\varepsilon)}{x\mu-1-\varepsilon} > 0.$$

Then from (43), (44), (79) and (80), we have

$$\begin{aligned} v(r, g) &< (\log M(r, g))^{1+\varepsilon} \leq (\log M(r, \beta))^s \\ &\leq (v(r, \beta) \log r)^s \leq v(r, \beta)^s v(r, g)^{s/x}, \end{aligned}$$

where $s = (1 + \varepsilon)/\mu$, and furthermore

$$v(r, g) \leq v(r, \beta)^y, \quad r \in U \setminus F.$$

ACKNOWLEDGEMENT

We are thankful to Dr. W. Bergweiler for his assistance in sending us a copy of A. A. Goldberg's paper (in Russian) and telling us its main idea. Also, we want to thank the referee for suggesting clarifications of some statements and the proof of Theorem 4.1. The second author wishes to express his appreciation for the hospitality extended by HKUST during his two month's visit there.

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(Received February 1, 1995 and in revised form December 13, 1995)