Financial Constraints, Hurdle Rates, and Investment Activity: Implications From a Multi-Period Model*

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Abstract

When a firm is financially constrained, an increase in its liquid balances (or net worth) is supposed to have a positive impact on its investment (equivalently, lower the hurdle rate for its projects). However, in a multi-period setting, an increase in liquid balances may make a firm more conservative in its choice of current projects. A firm with higher liquid balances today can carry its liquid balances to the next period. Thus, the firm is more willing to pass up a project today because it faces a lower risk of being constrained next period, and not being able to invest profitably. In a two-period setting, there is a critical level of liquid balances such that the firm's hurdle rate initially decreases in its liquid balances up to this level, then increases, before decreasing again. The non-monotonic behavior of the hurdle rate is also possible in a more general multi-period setting. For some special cases, the hurdle rate is non-monotonic every period in the level of liquid balances. These results are consistent with a variety of empirical evidence, such as the non-monotonic behavior of the cash flow sensitivity of investment with respect to the liquidity position of firms (Kaplan and Zingales (1997), Cleary (1999, 2002)), or the failure of Japanese small firms to step up their investment in spite of a steady build up of liquidity since the mid-nineties. The results also suggest that monetary policy may not be very effective in stimulating aggregate demand through the balance sheet channel, and may in fact have perverse effects, if firms have already accumulated cash balances.

JEL Classification Codes: E0, E3, E5, G3, G32.

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1 Introduction

A large literature has developed based on the idea that an increase in liquidity or net worth has a positive impact on the investment levels of financially constrained firms. In this paper, we argue that in a multi-period context where firms have the option of allocating their liquidity \textit{inter-temporally}, such a presumption is not valid in general. If firms have more liquid balances and these liquid balances can be carried into future periods, then firms are less likely to be financially constrained in the future as well. Since the risk of not being able to take projects in the future - should they pass up on projects today - is lower, firms may become more conservative in their project choice today as their liquidity position improves. In other words, investment may \textit{decline} today (equivalently, the hurdle rate for projects increase) as firms' liquid balances increase. Such a perverse implication is consistent with empirical evidence. In Japan, small firms have steadily increased the ratio of cash plus deposits to capital stock since 1996, and indeed, the level was higher in the second quarter of 2000 than in the second quarter of 1992. Yet, this apparent improvement in their liquidity position has failed to spur investment. Papers studying the relationship between changes in liquidity and corporate investment behavior (mostly for US firms) document that the relationship is a non-monotonic function of the level of liquidity, which is consistent with the existence of a range of liquid balances over which the relationship between liquidity and hurdle rates is perverse. Thus, our results have important implications for empirical tests of the cash-flow sensitivity of investment, or about how monetary policy might affect aggregate demand through the so-called "balance sheet channel"\footnote{The balance sheet channel refers to the impact of monetary policy on firms' balance sheets, such as their liquid balances or net worth. For example, lower interest rates will reduce firms' short term interest expenses, thereby improving their net income. This is supposed to have a positive impact on the investment levels of financially constrained firms. See Fazzari, Hubbard and Petersen (1988), Bernanke and Gertler (1989, 1995), Gertler (1992), and Bernanke, Gertler and Gilchrist (1996, 1998), for expositions of how financial market imperfections affect the impact of monetary policy on firms' investment decisions and real sector activity. Several papers show how financial market imperfections can produce cycles in real sector activity. See, for example, Kiyotaki and Moore (1997) and Suarez and Sussman (1997).}.

We adopt a multi-period version of a standard "moral hazard" model
much like that in Holmstrom and Tirole (1997, 1998). An entrepreneur has the option of investing every period in a project, drawn at random from some distribution. All projects require exactly one dollar of investment, and the distribution from which projects are drawn, as well as the project "type" once it is drawn (prior to investment), are common knowledge. Instead of investing in the project, the entrepreneur can invest in an "unproductive project" that gives him a private benefit, but does not generate any cash flows. If the entrepreneur has insufficient stock of funds (liquidity), he is forced to raise capital from outsiders. However, outsiders understand that the entrepreneur's incentive to choose the unproductive project after raising the funds is increasing in the amount of his repayment obligation. As a consequence, an entrepreneur with insufficient liquidity will not be able to find outside funding for the project. If the entrepreneur's liquidity is sufficiently low, we show that an increase in liquidity will increase the ex-ante probability that the entrepreneur is able to invest (i.e., the NPV of the marginal project, or equivalently, the entrepreneur's hurdle rate for projects, will decrease) \(^2\). This is exactly how, in existing literature, an increase in liquidity is supposed to relax the binding financial constraints.

However, the entrepreneur also has the option of not investing in today's project, and waiting for a better draw next period. The cost of not investing today is that, even next period, the entrepreneur can draw a project for which his liquid balances are insufficient. In that case, the entrepreneur will be unable to invest. However, this possibility becomes less likely as the entrepreneur's current stock of liquid balances increases, since this stock can be carried over to the next period. Thus, if the stock of liquid balances increases sufficiently, the entrepreneur will become more conservative in his choice of projects today. For the two-period version of the model, we find that if the stock of liquidity exceeds a critical level, the entrepreneur is less likely to invest today as his liquid balances increase (i.e. the NPV of the marginal project, or equivalently, the entrepreneur's hurdle rate for projects, increases). Eventually, however, the hurdle rate again decreases as the stock of liquid balances becomes sufficiently high \(^3\).

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\(^2\)However, under some regularity conditions, the NPV of the marginal project will always be positive.

\(^3\)Our results do not depend on the binary nature of investment, nor the random nature of projects. We are able to show that the possibility of non-monotonicity remains even
Some of the results generalize to models with arbitrary finite number of periods. The result that is most robust is that in every period of a finite period model, the sensitivity of the probability of investment to changes in liquidity decreases discontinuously as the stock of liquidity exceeds a critical level. Given that financially constrained firms tend to accumulate liquid balances during recessions, such a result is significant from the point of view of monetary policy's impact on aggregate spending through the balance sheet channel. The possibility of a perverse relationship between the stock of liquidity and the probability of investment, however, remains when the model is extended to an arbitrary (finite) number of periods. We are able to establish the existence of the perverse relationship in some special cases, and numerical simulations illustrate the possibility even in the general case.

The rest of the paper is organized as follows. Section 2 outlines the model, while section 3 presents the results on the relationship between liquid balances and the hurdle rate. Section 4 considers extensions to an arbitrary but finite number of periods, and section 5 extends the analysis to allow for the level of investment to be endogenously determined. Section 6 concludes.

2 The Model

We consider a two period model with three dates, \( t = 1, 2, 3 \). At time \( t = 1 \), an Entrepreneur (henceforth denoted by \( E \)) with a cash endowment of \( c \) can take a project. The project requires an investment of 1 unit of money and generates a cash flow of \( a > 0 \) with a probability \( p \) and 0 with probability \( 1 - p \). A is drawn from a continuous distribution \( F(a) \) with support \([a, \bar{a}]\) and a continuously differentiable density \( f(a) \) in the interior of the support. We will refer to \( a \) as the "state of the project". At the time \( E \) seeks outside financing, \( a \) is common knowledge. The cash flows from this project are realized at the end of the period, i.e. at \( t = 2 \).

At the beginning of the second period \( (t = 2) \), \( E \) has an opportunity to take a new project, again after a new draw of the state \( a \). This new draw is when the level of investment is continuous and chosen by the entrepreneur, and the state of the project is deterministic and identical in both periods.
assumed independent and identically distributed as the draw at \( t = 1^{4} \). Cash
flows from this investment are realized at \( t = 3 \).

In both these periods, \( E \) has the option of investing in an alternative
project that also requires investment of 1 unit of money. This project does
not generate any verifiable cash flow but gives \( E \) private benefit of \( B < 1 \).
We will refer to this alternative project as the "unproductive project" and \( E \)
will be said to invest "unproductively" if he chooses this project. Existence
of this alternative investment project is the source of "moral hazard" and
the resulting imperfection in the credit market.

We make the following assumptions:

A1. \( p\bar{a} - 1 > 0 \).
A2. \( pa - 1 \leq 0 \).

Let us define \( a^* = \frac{1}{p} \). Thus, a project has positive net present value if
and only if \( a > a^* \).

In any period, if \( E \) raises external funds to finance his project, then he
issues debt. For now, we will restrict the analysis to simple debt contracts in
which period \( t \) debt holders only have claims to cash flows realized in period
\( t + 1 \). This implies in particular that debt holders financing the project at
\( t = 1 \) only have a claim on the cash flows at \( t = 2 \), i.e. from the project
undertaken at \( t = 1 \). We will later consider debt contracts that allow period
1 debt holders to be paid, in the event of default, cash flows from the period
2 project as well. It will be shown that the simple debt contract analyzed
here is indeed optimal in this larger class of debt contracts.

To understand \( E \)'s investment decision in this dynamic setting, we start
as usual with the last period. Let \( x < 1 \) be the amount of cash that \( E \) has at
its disposal at \( t = 2 \) (after meeting his first period debt obligations, if any)
and let the state \( a \) be realized at \( t = 2 \). For \( E \) to invest in this state, he
must be able to raise additional debt of \( 1 - x \). For lenders to be willing to
lend, the following condition needs to be satisfied. At the face value of debt
that makes lenders break even should \( E \) invest, \( E \) must prefer, after raising
\( 1 - x \) from lenders, to invest in the project rather than invest unproductively.

\(^4\)It will be apparent that only the assumption that the draws are independent is sub-
stantive; the distribution of the first period draw of \( a \) has no major role in the analysis.
Hence, we must have

\[ p(a - d) \geq B \]  \hspace{1cm} (1)

where we assume that

\[ pd = 1 - x \]  \hspace{1cm} (2)

and \( d \) is the face value of the second period debt.

At this point, it is useful to point out that the moral hazard that we are modeling can be given alternative interpretations. For example, suppose that \( E \) can choose between two levels of effort: high and low, and \( B \) represents \( E \)'s utility from shirking. Shirking (choosing the low effort) causes the project to fail with certainty, while choosing the high effort causes it to succeed with probability \( p \). This is, in fact, the model that Holmstrom and Tirole (1997, 1998) use.

A less obvious, but important, interpretation is the standard risk-shifting behavior. Here, we can also think of \( E \) as a manager who is managing the firm in the interest of its owners. The conflict of interest associated with risk-shifting behavior is between owners and lenders. Suppose that \( E \) can choose an alternative project that pays an amount \( A \) with probability \( q \) and zero with probability \( 1 - q \). Let \( B = q(A - d) \), where \( d \) is the promised payment to debt holders. Make \( q \) very small and \( A \) very large so that \( p(a - d) < q(A - d) = B \) holds. If the alternative project is chosen, debt holders (almost) never get paid (since \( q \) is very small), but equity holders prefer this project. Thus, the model we propose is not limited to owner-managed firms.

Using the break-even condition for the leader, we can rewrite condition (1) as

\[ p(a - \frac{1 - x}{p}) \geq B. \]  \hspace{1cm} (3)

It is clear from condition (3) that if \( E \) is able to invest in state \( a \), then he is able to invest in all states \( a' > a \). For a particular \( x \), define \( a_x \) as the
marginal state in which E invests (in other words, E would be able to invest in period 2 if and only if \( a \geq a_x \)). Then we have:

**Proposition 1.** If \( \frac{B+x}{p} \leq \bar{a} \), then for all \( x \in [0, B) \), \( a_x \) is a strictly decreasing function of \( x \) given by

\[
 a_x = \frac{B}{p} + \frac{1-x}{p}.
\]

For \( x \geq B \), \( a_x = a^* \).

**Proof:** Notice that the infimum of \( a \) for which condition (3) holds is given by \( a_x \) as defined in equation (4). The proof is now immediate from equation (4) and the fact that since E can get \( x \) from not investing, he will never invest in state \( a < a^* \) (in which case his payoff is \( pa - 1 + x < x \)). □

Notice that since \( B > 0 \), \( u_0 > a^* = \frac{1}{p} \), i.e. the firm is not able to invest in all states (even if some of these states have positive net present value of investment) at \( t = 2 \), if it has no cash at \( t = 2 \). It is in this sense that a firm with no or limited cash in the second period is financially constrained.

If \( \frac{B+x}{p} > \bar{a} \), E is not able to invest at all for \( x \) satisfying \( \frac{B}{p} + \frac{1-x}{p} > \bar{a} \). Define \( a_x = \bar{a} \) if \( \frac{B}{p} + \frac{1-x}{p} > \bar{a} \), i.e. \( x < B - 1 - p\bar{a} \). The ex-ante (i.e. prior to the draw of the second project at \( t = 2 \)) payoff to E from having \( x \) units of cash in period 2 is

\[
P(x) = \begin{cases} 
\int_{a_x}^{\bar{a}} p(a - \frac{x}{p})dF(a) + F(a_x)x \\
\int_{a_x}^{a} [pa - 1]dF(a) + x.
\end{cases}
\]

Define \( \eta = \max(0, B + 1 - p\bar{a}) \).

**Observation 1.** (i) If \( \eta = 0 \), \( P'(x) > 1 \) for \( x \in [0, B) \) while \( P'(x) = 1 \) for \( x \geq B \). (ii) If \( \eta > 0 \), then \( P'(x) = 1 \) for \( x \in [0, \eta] \), \( P'(x) > 1 \) for \( x \in [\eta, B) \) while \( P'(x) = 1 \) for \( x \geq B \).

**Proof:** Follows immediately from the definition of \( P(x) \) and the fact that for \( x \geq B \), \( a_x \) is independent of \( x \). □
We now turn to the $t = 1$ investment choice of $E$. Suppose that state $a$ is realized. If $E$ raises $1 - c$ by issuing debt with face value $D$ and invests productively, then, the debt holders get back $D$ if $a$ is realized, however, if 0 is realized, they get back nothing. Thus, for the lenders in period 1 to break even, we must have

$$pD = 1 - c.$$  \hspace{5cm} (6)

Furthermore, it must be in the interest of $E$ to invest productively. This requires that

$$pP(a - D) + (1 - p)P(0) \geq B - P(0). \quad (IC)$$  \hspace{2cm} (7)

Condition (7) will be called the Incentive Compatibility (IC) constraint. Moreover, $E$ can always decide not to invest and carry over the cash $c$ to next period. Thus, we must have the following Individual Rationality condition:

$$pP(a - D) + (1 - p)P(0) \geq \mathcal{P}(c). \quad (IR)$$  \hspace{2cm} (8)

The IC condition simply requires that $E$ does not invest in the unproductive project, while the IR condition ensures that $E$ is better off investing in state $a$ than not investing and carrying its initial cash holding $c$ to the next period.

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1We assume that if $E$ borrows, and carries over any cash into the second period, then debtholders can claim that cash in the event of default. This assumption implies that $E$ will not borrow more than $1 - c$. In a working paper, we allow the possibility that the entrepreneur and lenders at $t = 1$ agree that the former can carry over and set aside an amount at $t = 2$ even in the event of default. The qualitative results remain unchanged. One justification for the present assumption is as follows. Suppose that the entrepreneur has existing assets that will generate cash flows $\phi$ at $t = 2$. It can be shown that if the entrepreneur can pledge $\phi$ to $t = 2$ lenders in the event of default, then effectively, he can invest as though he had cash $c + \phi$ at $t = 1$. However, $\phi$ may not be verifiable, so that the entrepreneur may have to allow the $t = 1$ lenders to claim everything if the period 1 project fails. Any cash carried over from the first period will also be claimed by debtholders in this case. Our analysis would remain unchanged, with $c$ replaced by $c + \phi$, and the entrepreneur will not borrow more than $1 - (c + \phi)$. 

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3 Liquid Balances and the Hurdle Rate

We now establish some results that illustrate how the Incentive Compatibility and Individual Rationality constraints come into play and affect E's investment behavior at $t = 1$.

To do so we first define a critical value of $c$ that will play an important role in our subsequent analysis.

Let $c^0$ solve

$$P'(c^0) = P(0) + B$$

(9)

Claim 1. $c^0$ exists, is unique, and satisfies $0 < c^0 < B$.

Proof. At $c = 0$, $P(0) < P(0) + B$. Further at $c = B$, by Proposition 1, E can invest in all productive states, i.e., $a_c = a^*$. Thus $P(B) = B + \int_{a^*}^{B}(pa - 1)dF$. Hence, $P(B) - B > P(0)$ (since by A1, $a > \frac{1}{p} = a^*$ and $a_0 > a^*$). Thus, by continuity, there exists a $c^0 < B$ such that $P(c^0) = B + P(0)$. Since $P(.)$ is increasing, $c^0$ is unique.

Lemma 1. If E with aggregate cash holding $c$ invests in state $a$ at $t = 1$, then he must invest in all states $a' > a$.

Proof. The proof immediately follows from the fact that if the IC and IR constraints holds for $a$, they hold for all $a' > a$.

By virtue of Lemma 1, one can define for any cash holding $c$, the marginal state in which E invests at $t = 1$ as the infimum of all $a$ for which E invests, i.e. the IC and IR conditions both hold. We shall denote this marginal state by $c^6$.

Lemma 2. (Failure of Intertemporal Equality of Returns). For all $c < c^0$, in any state in which E invests in period 1 (including the marginal state), his payoff is strictly greater than $P(c)$, i.e., the IR constraint does not bind.

Proof: If E invests at $t = 1$ in state $a$, then, he can always invest in the unproductive project and obtain a payoff of $B + P(0)$. By not investing in

\footnote{We use superscripts to denote marginal states at $t=1$ and subscripts to denote marginal states at $t=2$.}
period 1, E can get at most $P(c)$. Since $c < c^0$, $P(c) < P(c^0) = B + P(0)$, the result follows.

Lemma 2 shows an important property of E's investment behavior. If E has insufficient cash in period 1, then he is unable to equate the inter-temporal rates of return on his investment projects. The marginal project in which he invests today has a strictly higher return than the alternative of not investing today and investing tomorrow. The reason for the failure of inter-temporal equality of rates of return is that, even though a project available today (corresponding to some $a^c$ slightly below $a^c$) may be more profitable than the alternative of drawing a project randomly and investing next period, if E has insufficient cash, he may not be able to undertake today's project because the IC constraint will not be satisfied.

We are now ready to address the issue of E's choice of projects. An important issue related to financial constraints is how the firm's hurdle rate for today's projects is affected by its current liquidity. Notice that if the state of the marginal project $a^c$ is decreasing in $c$, then the firm will accept a project that it would have rejected at a lower level of $c$, which implies that the hurdle rate decreases in $c$. Is the firm's hurdle rate necessarily decreasing in its aggregate cash holding $c$? We address this now.

**Proposition 2.** For any $c < c^0$, we must have $a^c > a^c^*$ if $c < c^*$, i.e. the marginal state (and hence the hurdle rate for current projects) is decreasing in $c$.

To prove the Proposition, the following Lemma will be used.

**Lemma 3.**

(i) For all $c < c^0$, the IC constraint binds at the marginal state $a^c$ (and the IR constraint does not bind).

(ii) For all $c > c^0$, the IR constraint binds at the marginal state $a^c$, and the IC constraint does not bind.

**Proof:** Please see the Appendix.
Proof of Proposition 2: For $c < c^0$, we know from Lemma 3 (part (i)) that the IC constraint binds in the marginal state. Thus for any $c < c^0$, we have

$$p[P(a^e - (1 - c)/p) - P(0)] = B.$$ 

If $c$ increases $(1 - c)/p$ falls and thus $a^e$ must fall as well.

At $c = c^0$, Lemma 3 indicates that the binding constraint for the marginal state switches from the IC constraint to the IR constraint. This has immediate implications for the sensitivity of the marginal project to changes in liquid balances. The following is a robust result in the sense that it continues to hold when the model is extended to any (arbitrary) finite number of periods, as we shall see later.

Proposition 3. The slope of the $a^e(c)$ function changes discontinuously at $c = c^0$, and $\lim_{c \to c^0} \frac{da^e}{dc} > \lim_{c \to c^0} \frac{da^e}{dc}$. \footnote{It can be shown that the slope of the $a^e$ function is more negative for $c < c^0$ than for $c > c^0$, except possibly over the interval $c \in [1, 1 - B]$.}

Proof. Please see the Appendix.

Proposition 3 indicates that the impact of monetary policy through the balance sheet channel (an increase in $c$) could be muted if firms' current cash balances exceed $c^0$. We postpone a more complete discussion of this and related issues until later. For now, we show that, in the two-period setting, we can in fact get a much stronger result regarding the way a change in $c$ affects the marginal project (equivalently, E's hurdle rate for projects), namely, the hurdle rate increases as E's cash holdings increase beyond $c^0$. The robustness of this particular result to extensions of the model to any arbitrary but finite number of periods will be discussed later.

To prove this result, we will first make use of the following curvature assumption on $P(.)$. This curvature assumption has an appealing implication that is consistent with the analysis of Froot, Scharfstein and Stein (1993), but is not really needed for our main results. The curvature assumption will be relaxed later.

A3. $P(x) > x$ implies that $P(x')$ is concave on $[x, \infty)$. Moreover it is strictly concave for $x < x' < B$. 

Remarks

- Notice that by virtue of Observation 1, $P(.)$ can at best be weakly concave over $[0, \infty)$, since for $x > B$, it is linear.

- $P(x)$ is not necessarily concave on the entire range $[0, B]$ if $P(0) = 0$. See Figure 1 for a graph of $P(.)$ when $P(0) = 0$.

- If $P(0) > 0$, it is easy to show that $P(x)$ is strictly concave on $[0, B]$ when $F(a)$ is uniform. See Figure 2 for a graph of $P(.)$ when $P(0) > 0$.

- The assumption of concavity of the $P(x)$ function is similar to the assumption of concavity of the profit function in Froot, Scharfstein and Stein (1993). In their framework, the concavity of the profit function provides a motive for hedging cash flows. In fact, it can be shown that if $P(x)$ is strictly concave over $[0, B]$, then E will prefer not to invest in the zero NPV (risky) project at $t = 1$ which corresponds to $a = a^* = \frac{1}{p}$.

Proposition 4. Under A3, $a^*$ is strictly increasing in $c$ in the interval $[c^0, B]$.

Proof: From Lemma 3 (ii), we know that IR constraint must bind at $c > c^0$ at the marginal state $a^c$. Letting $c' = a^c = \frac{k - c}{p}$, we thus have,

$$pP(c') + (1 - p)P(0) = P(c).$$

Totally differentiating the IR constraint, one gets

$$pP'(c') \frac{da^c}{dc} + 1/p = P'(c)$$

or

$$pP'(c') \frac{da^c}{dc} = P'(c) - P'(c')$$

From the IR constraint, it is clear that $c' = a^* = \frac{k - c}{p} > c$. Two cases can arise.

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<sup>8</sup>Here, $F(a)$ is uniform over $[\frac{1}{p}, 3]$, $p = 0.5$ and $B = 0.75$. It can be shown that concavity is satisfied for all $c \in [c^0, B]$ if $F(c)$ is uniform.

<sup>9</sup>This case corresponds to $\eta = 0$ - see the definition of $P(x)$.

<sup>10</sup>Here, $F(a)$ is uniform over $[\frac{1}{p}, 4]$, $p = 0.5$ and $B = 0.75$. 

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Case 1. \( c' \geq B \).

From Observation 1, we then have \( P'(c') = 1 \) and \( P'(c) > 1 \) (since \( c < B \)), and \( c \) must exceed \( \eta \), as \( c > c' \) and \( P(c') = B + P(0) > c' \). From equation (10), it then follows that \( \frac{dc}{dx} > 0 \).

Case 2. \( c' < B \).

Since \( P(c') > P'(c') \) for \( c < c' \). From equation (10) we then have \( \frac{dc}{dx} > 0 \).

Notice that the slope of the \( \alpha(c) \) function becomes flat for \( c \in (B, 1) \), since both \( P'(c) \) and \( P'(a - \frac{1-c}{p}) \) are unity. However, for \( c > 1 \), the IR constraint is

\[
pP(a - c - 1) + (1 - p)P(c - 1) = P(c).
\]

Differentiating, and using Observation 1 and the fact that from the IR constraint, we have \( a^c + c - 1 > c > 1 \), one gets

\[
p\left(\frac{da}{dc} + 1\right) + (1 - p)P'(c - 1) = 1.
\]

For the \( P(0) > 0 \) case, \( P'(c - 1) > 1 \) for \( 1 + B > c > 1 \), and hence \( \frac{dc}{dx} < 0 \). For the \( P(0) = 0 \) case, \( P'(c - 1) = 1 \) and \( \frac{dc}{dx} = 0 \) for \( 1 < c < B + 1 - (p\eta - 1) \), and \( P'(c - 1) > 1 \) and \( \frac{dc}{dx} < 0 \) for \( 1 + B > c > B + 1 - (p\eta - 1) \). It is easy to check that \( a^c \) will converge to \( a^* = 1/p \) as \( c \) approaches the value of \( 1 + B \).

3.1 The Relevance of the Non-Monotonicity Result

3.1.1 Cash Flow Sensitivity of Investment.

A large literature finds that firms that are less financially constrained (such as larger firms, firms with business group affiliations, or those with debt and commercial paper ratings) show less sensitivity of investment to changes in internal funds or liquidity than more financially constrained firms (see Hubbard, (1998) for a survey). This issue is important for several reasons. First, it shows the relevance of information problems firms face vis-a-vis external
providers of finance for their financing and investment decisions. Thus, an implication of the existence of such problems is that financial development will affect the growth rates of firms by reducing the extent of these information problems (Rajan and Zingales, 1998). Another implication, already discussed in the introduction, is that monetary policy, by affecting firms’ balance sheets and their liquidity positions, could affect real sector activity.

Recent studies, however, have found some puzzling empirical regularities about the cash flow sensitivity of investment. For example, Kaplan and Zingales (1997) and Cleary (1999, 2002) find that the sensitivity of investment to cash flows is higher for firms with better liquidity positions at the beginning of a particular year than those with worse liquidity positions. Our results on how the marginal project responds to changes in the firm’s liquid balances (i.e. how $\frac{dc}{dc}$ changes with c) are appealing because they show that such a relationship is clearly possible. Cash flows from existing assets would increase the liquid balances held by firms. Holding the initial liquid balance c fixed, would those with higher cash flows invest more? For firms with initial c much below $c^0$, the larger the increase in c, the larger is the proportion of firms investing, since the marginal state $a^2$ decreases and the probability of investment $1 - F(a^2)$ increases in c for $c < c^0$. However, for firms with c closer to $c^0$, an increase in c may carry such firms beyond the threshold of $c^0$ and cause them to become more conservative in their choice of projects. Thus, investment does not respond as much to increases in liquidity or cash flows for such firms, i.e. the cash flow sensitivity of investment is low. However, as the initial stock of liquid balances increases further, firms are more willing to invest in response to increases in liquidity, as even after investing, they can carry over some cash into the next period. The cash flow sensitivity of investment is high for such firms.

Thus, in terms of our model, firms with intermediate liquidity (i.e. c around $c^0$) will exhibit lower cash flow sensitivity than those with high liquidity. This is consistent with the empirical evidence cited above; however, it remains to explain why, contrary to the implications of our model, the studies mentioned above do not find cash flow sensitivity to be high for firms with low initial liquidity. Both Kaplan and Zingales (1997, 2000) and Fazzari, Hubbard and Petersen (2000) agree that firms with low liquidity in their sample may actually be financially distressed firms that do not have much discretion in spending incremental cash flows. This may explain why firms
with very low liquidity do not exhibit much cash flow sensitivity. Alternatively, it could also be the case that the empirical sample of firms with low liquidity contains some financially unconstrained firms (i.e. those with low or zero $B$) that do not need to carry a lot of liquid balances as they will not be constrained in the future. This could prevent the cash flow sensitivity of investment from showing up for the firms that have the least liquid balance sheets.

Several other papers have addressed the issue of the non-monotonic relation between changes in liquidity and investment, although none explore the intertemporal issues that we stress in this paper. Kaplan and Zingales (1997) work in terms of a reduced-form model in which the cost function for external finance is increasing and convex, and show that the way in which the degree of financial constraints (a parameter in the cost function for external finance) affects the sensitivity of investment to changes in liquidity is, in general, ambiguous. Pervel and Raith (2001) show that investment is decreasing in liquidity if net worth is negative, but is increasing in liquidity if net worth is positive. Almeida and Campello (2001) use the framework of Hart and Moore (1994) to argue that an increase in liquidity will "relax" the constraints less for firms with greater scope for diverting the liquidation cash flows from investment (the "more constrained" firms). Since external lenders will recover a smaller proportion of these cash flows as they step up lending, more constrained firms will exhibit smaller sensitivity of investment to changes in liquidity.

3.1.2 Cash Holdings and Investment Behavior of Japanese Firms in the Post-Bubble Period

A number of explanations have been put forward to explain the sluggish investment behavior of Japanese firms in the post-bubble era, such as poor fiscal policy, liquidity trap, or a reaction to over-investment in the bubble period. One view is that Japanese firms have been experiencing a "credit crunch" because of Japanese banks' unwillingness or inability to lend. Hayashi and Prescott (2002) point out that the ratio of cash plus deposits to capital

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11The significance of intertemporal issues, however, is often raised in the literature. See Fazzari, Hubbard and Petersen (2000), page 701 and Gilchrist and Himmelberg (1998).
stock of small Japanese firms has been much higher (around 0.4) than for US firms (around 0.2) in the post-1992 period. Moreover, since 1996, this ratio has been steadily increasing for Japanese small firms. Thus, Japanese firms do not appear to be financially constrained. What, then, accounts for the sluggish investment behavior of Japanese firms?

Hayashi and Prescott (2002) focus on a slowdown in total factor productivity (TFP) growth as the primary reason for the "lost decade of growth". Our analysis in this section suggests that an improved liquidity position - per se - does not imply that financial constraints have no bearing on the investment behavior. If firms anticipate being constrained in the future, then they may be more inclined to pass up on investment opportunities today as their liquidity positions improve. A deterioration in investment opportunities, or reaction to overinvestment in the bubble period, may have caused Japanese firms to postpone investment after the collapse of the bubble. As a result, their liquidity positions may have been improving since the collapse. However, with more liquidity in their balance sheets, Japanese firms may have become more conservative in their choice of projects as well. A combination of these two factors is consistent with the lack of investment activity and a build-up of liquidity position since the mid-nineties.

3.2 Relaxing the Concavity Assumption on $P(x)$.

The concavity assumption A3 is appealing because it is consistent with a hedging motive, as well as the empirical results discussed above. Nonetheless, it is not required for the non-monotonicity result. We now proceed to show this.

Notice that in the proof of the Proposition 4, the concavity assumption is not invoked in Case 1, i.e. if the cash flow of the project following success exceeded $B$. This immediately suggests that one can prove a stronger result that will generate non-monotonic behavior even when no curvature restrictions are imposed on $P(\cdot)$. To do so let us define

$$C = \{c | a^c - \frac{(1-c)}{p} \geq B\}.$$ 

It is easy to check that $C$ is non empty. Let $\bar{c}$ be the infimum of $C$. 

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Claim 2. $\bar{c} < B$.

Proof: At $c = B$, if $E$ invests in any state $a$, by IR we have

$$pP(a - (1 - B)/p) + (1 - p)P(0) \geq P(B).$$

Since $P(B) > P(0)$, $a - (1 - B)/p > B$. The result now follows from continuity of $P(.)$

Let $c^* = \max\{c^d, \bar{c}\}$.

Proposition 5. $a^c$ is strictly increasing in the interval $[c^*, B]$.

The following Lemma will be required

Lemma 4. If $B > c > c^*$, then $c \in C$.

Proof. Please see the Appendix.

Proof of Proposition 5: Since $c > c^* \geq c^d$, the IR constraint must bind at the marginal state $a^c$. Totally differentiating this constraint one gets

$$pP'(c') \frac{da^c}{dc} = P'(c) - P'(c')$$

where $c' = a^c - (1 - c)/p$. By Lemma 4, $c' \geq B$ and thus by Observation 1, $P'(c') = 1$. Since $c < B$ and $c > c^d > \eta$, using Observation 1 once more we get $P'(c) > 1$, and the Proposition follows.

Our analysis thus far has assumed that $E$ uses a simple one period debt contract. We end this section by showing that if the initial cash holding $c$ exceeds $c^* \geq c^d$, then the one period debt contract is indeed optimal in the class of all contracts. This, in particular, implies that Proposition 5 will continue to hold when a larger set of debt contracts is allowed.

3.3 Optimality of the Simple Debt Contract

Recall that in a one period debt contract the debt holders get back zero if the project is a failure. However, one can argue that the first period debt holders can be paid from the cash flow that will be realized if the second
period project results in a success. Since the second period project a will be common knowledge, it will also be possible to make such debt payments conditional on the realization of a.

Formally one can define a debt contract by a pair \( M = (D^*, D(a))_{a \in A} \) where \( D^* \) represents the payment to first period debt holders if period 1 project is a success while \( D(a) \) represents the payment to the same debt holders when the first period project is a failure and the state of the second period project is \( a \).

We impose the following restrictions on \( M \). \( M \) is said to be feasible if

- For every \( a \), \( D(a) \geq 0 \).
- If \( D(a) > 0 \), then \( p[a - 1/p - D(a)] \geq B \).

A non-negativity restrictions on \( D(a) \) seems natural, while the second restriction is simply the requirement that if debt holders have to get back \( D(a) \), then, it must be incentive compatible for \( E \) to invest productively in period 2.

Given a feasible contract \( M = (D^*, D(a))_{a \in A} \), let \( D(M) \) (resp. \( Y(M) \)) be the expected payoff to the first period debt holders (resp. \( E \)) if the first period project results in a failure. The fair pricing of debt requires that

\[
pD^* + (1 - p)D(M) \geq 1 - c
\]

and the IC and the IR constraints for \( E \) in period 1 with cash holding \( c \) and the realized state \( a \) are given by

\[
pP(a - D^*) - (1 - p)Y(M) \geq B + Y(M)
\]

\[
pP(a - D^*) + (1 - p)Y(M) \geq P(c)
\]

Note that a one period debt contract \( M \) has the property that \( D^* > 0 \) and \( D(a) = 0 \) for all \( a \). We now show that if the cash holding of \( E \) is above \( c^* \), then the optimal debt contract is in fact this simple contract.

**Proposition 6.** If \( c \geq c^* \), then the simple debt contract, \( M = (D^*, 0)_{a \in A} \) is an optimal debt contract.

**Proof:** Please see the Appendix.
4 Many Periods

In this section, we extend our model to allow for more periods. One way to justify a model with a terminal period is to appeal to the fact that many entrepreneurial firms sell out shortly after going public. If the new buyer is a large firm that can be considered financially unconstrained, then the objective of E (or the manager of the firm - see the alternative interpretation of the moral hazard problem in section 2) is to maximize the value of the firm’s terminal period $T + 1$ cash accumulated from taking projects from time $t = 1$ to $t = T$.

So, let us consider any $T$ and assume that every period, a state $a$ will be realized and E will have the option to invest. Assume that E maximizes the undiscounted cash holding of the end period $T + 1$. Moreover, in any period, if he decides to borrow, he does so using a simple one period debt contract. Denote by $V^t(x)$ the expected payoff to E at period $t$, $t \leq T$ when his cash holding is $x$. Clearly, $V^T(x) = P(x)$. Suppose now we have determined $V^t(x)$ for all $t = T, T - 1, \ldots, (T - \tau + 1)$ and let $t = T - \tau$. Let $x$ be the cash holding of E at $t$.

First consider $x > 1$.

Given $a$, if E invests in state $a$, then define its payoff for $t < T$ as

$$V^t_I(a, x) = pV^{t+1}(a + (x - 1)) + (1 - p)V^{t+1}(x - 1).$$

If he does not invest, then, define

$$V^t_N(a, x) = V^{t+1}(x).$$

Finally, define $V^t(a, x) = \max\{V^t_I(a, x), V^t_N(a, x)\}$ and define

$$V^t(x) = \int_a V^t(a, x) dF(a).$$

Now assume that $x < 1$. If E has to invest in $a$, he has to raise $(1 - x)$. Moreover after raising this amount, E must invest productively. Thus, the

\[12\text{In our analysis, we have assumed that the discount rate is zero. If the financially unconstrained buyer were to operate the firm for ever, a discount rate would have to be introduced. However, this should not alter the substance of our analysis in any important way.}]}
following IC constraint must be satisfied

\[ V_t^i(a, x) = pV_{t+1}^i(a - (1 - x)/p) + (1 - p)V_{t+1}^i(0) \geq B + V_{t+1}^i(0) \quad \text{(IC)} \]

Moreover, \( E \) will invest in \( a \) only if the IR condition is satisfied, i.e., if

\[ V_t^i(a, x) \geq V_N^i(a, x) = V_{t+1}^i(x). \quad \text{(IR)} \]

Thus, define, \( a^i_c \) as the infimum of the set of \( a \) that satisfy the IC and IR constraints, and define

\[ V^i(x) = \int_{a^i_c} V_t^i(a, x) dF(a) + F(a^i_c)V_{t+1}^i(x). \]

For any \( t < T \), let \( c^i \) satisfy the following equation

\[ V^{t-1}(c^i) = B + V_{t+1}^i(0). \]

Claim 3. For every \( t < T \), \( c^i \) exists and is unique. Moreover, \( c^i < B \).

Proof: To prove this claim, we use the following Lemma, which is proved in the Appendix.

**Lemma 5.** For \( x, y \geq 0 \) with \( x > y \) \( V^i(x) - V^i(y) \geq (x - y) \) with strict inequality if \( y < B \).

We now prove Claim 3. Fix \( t < T \). Then at \( c = 0 \), \( V_{t+1}^i(0) < B + V_{t+1}^i(0) \). Further at \( c = B \), by Lemma 5, \( V_{t+1}^i(B) - V_{t+1}^i(0) > B \). Since \( V_{t+1}^i \) is continuous in \( c \), there exists \( c^i < B \) such that \( V_{t+1}^i(c^i) = B + V_{t+1}^i(0) \). Since \( V_{t+1}^i \) is a strictly increasing function of \( c \) (Lemma 5), \( c^i \) is unique.

Let \( a^i_c \) denote the marginal state at which \( E \) with cash holding \( c \) invests in period \( t \).

**Lemma 6.** Fix \( t \) and \( c \).

- If \( c < c^i \), the IC constraint binds at \( a^i_c \) and the IR constraint is slack.
• If \( c > c' \), the IR constraint binds at \( a_c' \) and the IC constraint is slack.

Proof: The proof follows exactly that for Lemma 3.

The result that is most robust to the extension to this multi-period setting is the analogue of Proposition 3, i.e. the fact that the slope of the marginal state \( a_t'(c) \), for any \( t \), changes discontinuously at \( c' \).

**Proposition 7.** The slope of the \( a_c'(c) \) function changes discontinuously at \( c = c' \), and \( \lim_{c \downarrow c'} \frac{\partial a_t'}{\partial c} > \lim_{c \uparrow c'} \frac{\partial a_t'}{\partial c} \).

Proof: The proof is completely analogous to that for Proposition 3.

In other words, monetary policy aimed at stimulating aggregate demand through the balance sheet channel will necessarily have a muted effect if the stock of liquid balances already exceeds a critical level. Unfortunately, we cannot use the method of proof as in Proposition 4 to show that it must also necessarily have a perverse effect for some level of liquid balances, since it can be shown that the \( V^t(x) \) function is not concave in general. However, as is apparent from the proof of Proposition 5, concavity is a rather strong sufficient condition for non-monotonicity. Figure 3 shows that, for a particular example with \( T = 4 \), non-monotonicity is exhibited in every period except the last.

Non-monotonicity, however, can be shown to hold for every \( t < T \) in some special cases, or more correctly, variants of the model. Below, we give two such examples.

Example 1. Suppose that the firm will be sold immediately to a financially unconstrained buyer any time \( E \) invests from \( t = 1 \) to \( t = T \) and is successful (with probability \( p \)). If \( E \) invests but fails, or if \( E \) does not invest in any period, then it continues to be “active” (in the sense that next period it draws an \( a \) from \( F(a) \), but may or may not invest) until either it is successful, or is liquidated/sold at \( t = T + 1 \).

\[ \text{For non-monotonicity, what is required is that for some } c > c', \left. \frac{\partial V^{t+1}(c)}{\partial s} \right|_{s = c'} < \left. \frac{\partial V^{t+1}(s)}{\partial s} \right|_{s = c}, \text{ where } c' = a_t' - \left(1 - c\right)/p. \]

\[ \text{For Figure 3, } F(a) \text{ is uniform over } [\frac{1}{2}, 2], p = 0.75 \text{ and } B = 0.75. \]
Example 2. Suppose that if E invests and fails, then it does not invest any more. If it is successful or does not invest, it remains active. The assumption that an E that invests and fails never invests again could be motivated by reputational issues. However, we are not invoking default-related liquidation, since the assumption would need to hold even when E has \( c > 1 \) and does not need to borrow. Our justification is that if a project fails, then the firm may lose customers, suppliers and skilled workers. Notice that for our model to be valid, we only need the creditors and E to be symmetrically informed about the firm’s success prospects. Other stakeholders may be asymmetrically informed, and failure may scare them off.

Proposition 8. For either examples 1 or 2, an active firm in every period \( t < T \) exhibits non-monotonicity of the hurdle rate.

Proof: Please see the Appendix.

To sum up our results in this section, even in a finite period model, exogenous changes in firms’ cash balances - for example, due to the impact of monetary policy - may have muted or even perverse real effects, especially when the firms are already carrying sufficient liquidity in their balance sheets.

5 Endogenous Investment

In this section, we allow for the investment level to be chosen by E. Unlike the case considered thus far (where the level of investment is fixed), we here show that the first period level of investment can indeed be non-monotonic in the cash holding \( c \) even when there is no uncertainty with respect to the state of the project.

To show this, we will assume that the state of nature \( a \) is the same in both periods and thus subsume it in the payoff function. For any investment level, the project is successful with probability \( p \) in which event, E earns a payoff of \( \pi(I) \) (gross of any debt liabilities). With probability \( (1-p) \), however, the project fails and the return is zero. We assume that \( \pi(I) \) is a strictly concave function of \( I \), and \( \lim_{I \to 0} \pi(I) > \frac{1}{p} \). We also assume that the return from investing \( I \) in the unproductive project is given by \( BI \), where \( 0 < B < 1 \).
We first consider the investment behavior of $E$ in period 2 when he has cash amount $c$ at the beginning of that period. Let $I^*$ maximize $p\pi(I) - I$. For $c < I^*$, denote by $I(c)$ that level of $I$ that solves

$$p\pi(c + I(c)) - I(c) - B(c + I(c)) = 0. \quad (14)$$

Since $\pi$ is strictly concave and $B < 1$, there is a unique $I(c) > 0$ for $c < I^*$.

The following Lemmas will be useful:

**Lemma 7.**

1. If $c > I^*$, $E$ invests $I^*$ and the payoff is $P(c) = p\pi(I^*) - I^* + c$.

2. If $I(c) + c \geq I^* > c$, then $E$ borrows an amount $z = I^* - c$ from the market and invests the entire amount $I^*$ in the productive project. The payoff to $E$ in such a case is given by $P(c) = p\pi(I^*) - I^* + c$.

3. If $I(c) + c < I^*$, then $E$ borrows $I(c)$ from the market, invests $I(c) + c$ in the productive project and obtains a return of $P(c) = p\pi(c + I(c)) - I = B(I(c) + c)$.

**Proof.** It is clear that in any equilibrium $E$ will never invest in the unproductive project. If $I$ is the amount that $E$ borrows in the market, then fair pricing debt of face value $D$ requires $D = I/p$, and incentive compatibility requires that $p[\pi(c + I) - D] = p\pi(c + I) - I \geq B(c + I)$. The Lemma then follows immediately.

**Lemma 8.** $c + I(c)$ is increasing in $c$ for $c < I^*$.

**Proof.** Differentiating equation (14), we get

$$1 + \frac{dI(c)}{dc} = \frac{1}{(1 + B) - p\pi'(c + I(c))}.$$

Now notice that the function $p\pi(c + I) - (B + 1)I - Bc$ is strictly positive at $I = 0$ since $p\pi(c) - Bc > 0$ for $c \leq I^*$. Since $\pi(.)$ is concave, it follows that $p\pi(c + I) - (B + 1)I$ must be decreasing in $I$ at $I = I(c)$. Thus, $p\pi'(c + I(c)) - (1 + B) < 0$ and hence $1 + \frac{dI(c)}{dc} > 0$.

The following is now immediate from the Lemmas.
Corollary. Suppose \( I(0) < I^* \). There exists \( c^* \), where \( I^* > c^* > 0 \), such that

1. If \( c < c^* \), the payoff to \( E \) in period 2 is \( B(c + I(c)) \).
2. If \( c > c^* \), the payoff to \( E \) in period 2 is \( p\pi(I^*) - I^* + c \).

We now state the main result of this section.

Proposition 9.

1. If \( I(0) > I^* \), then for any \( c \), the first period investment is given by \( I^* \).
2. If \( I(0) < I^* \), the investment in the first period can be non-monotonic with respect to the first period cash holding \( c \).

Proof. The first part of the Proposition is straightforward. If \( I(0) > I^* \), then the lack of cash does not create any problem for \( E \) to invest the efficient level \( I^* \) in the second period. Further, one can show that if it is incentive compatible to borrow and invest \( I^* \) in period 2, then it is also incentive compatible for \( E \) to borrow any \( I \leq I^* \) in the first period and invest productively. Therefore the first period investment of \( E \) is independent of his cash holding and equals the efficient investment level \( I^* \).

For the second part of the Proposition, we use the following example.

Let \( \pi(I) = 4I - (1/4)(I^2) \), \( p = 1/2 \) and \( B = 3/4 \). It is easy to check that \( I^* \) for this problem is given by \( I^* = 4 \). Moreover, it can be checked that \( I(0) = 2 \) and thus in period 2, \( E \) with cash holding of zero will be able to invest only 2 units and get a return of \( 2B = 3/2 \).

Now consider \( E \) with \( c = 1 \) in the first period. We claim that \( E \)'s optimal choice of investment in period 1 is exactly \( I^* \) and \( E \) borrows 3 units from the market. The expected payoff to \( E \) from borrowing an amount \( z \) from the market and investing \((1 + z)\) in period one is

\[
p[\pi(I^*) - I^* + 4(1 + z) - (1/4)(1 + z)^2 - D] + (1 - p)2B
\]
provided the efficient amount $I^*$ can be invested in the second period. From Lemma 7, the term within square brackets represents the second-period payoff to E if he is successful in the first period (with probability $p$), in which case the cash at the beginning of the second period is $4(1+z) - (1/4)(1+z)^2 - D$. Clearly, $D = 2z$. The first order condition with respect to $z$ gives us

$$4 - (1/2)(1+z) - 2 = 0$$

Thus $z = 3$. Note that at $z = 3$, $4(1+z) - (1/4)(1+z)^2 - D = 6 > I^*$, thus E can invest the optimal amount in the second period. Moreover at $z = 3$, the payoff to E when the project succeeds is $8$. Thus the expected payoff from investing productively in the project is $(1/2)(8) + (1/2)(2B) = 4 + B$. If E invests in the unproductive project, E will receive $4B$ in the first period followed by $2B$ in the second period. The payoff to E will then be $6B$. Since $B = 3/4$, E is better off investing in the productive project.$^{15}$

Consider now E with a first period cash holding of $c = 4$. It can be shown easily that the optimal investment in period 1 cannot be more than $I^* = 4$. We show now that the optimal investment of E in period 1 must be strictly less than $I^* = 4$. If E invests $4$ in the first period, his expected payoff is

$$p[p\pi(I^*) - I^* + \pi(I^*)] + (1-p)2B.$$ 

Using the parameter values, the payoff is $31/4 = 62/8$.

Assume now he invests $3$ dollars and retains $1$ dollar for second period investment. If the project succeeds, the payoff to E is $p\pi(I^*) - I^* + \pi(3) + 1 = 51/4$. If the project does not succeed, E has $1$ dollar and it is easily checked from (14) that E will be able to borrow $3$ units in the second period without violating the incentive constraint. The payoff to E if the project fails is then $p\pi(I^*) - I^* + 1 = 3$. The expected payoff to E from investing $3$ dollars is then $(1/2)(51/4) + (1/2)(3) = 63/8$.

$^{15}$It is easy to check that E will not be better off by not borrowing and investing an amount less than or equal to 1. E's payoff in this case is strictly less than what it would be if (i) he invested exactly 1 dollar in period 1 (ii) invested the efficient amount of 4 dollars in period 2 if the project succeeded, and (iii) if the project failed, still had 1 in his hand, i.e. less than $p(y + p(4(4) - (1/4)(4)^2) - 4) + 1 - p)B(1 + I(I))$, where $y = 4 - (1/4)$. It is easily checked that $I(1) = 3$, and thus the last expression equals $4 + (3/8)$, which is less than $4 + B$.  

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Thus E with cash holding 4 units must invest less than $I^*$.  

The intuition for non-monotonicity is that since E can borrow, he can invest close to the efficient level in the second period even with a cash balance much less than that level. Thus, if he has enough cash to invest efficiently in period one, he may prefer to invest less than the efficient level and carry over enough cash into the second period so that he can invest a high amount in that period even if the period one project fails. On the other hand, if he does not have enough cash in period one, this option is not very attractive, and E would prefer to borrow and invest a larger amount in the first period so that, conditional on success, he can invest a larger amount in the second period. This is the basic reason for non-monotonicity.

6 Conclusion

When a firm is financially constrained, an increase in its liquid balances need not cause it to invest more immediately. Rather, an increase in liquid balances may make a firm more conservative in its choice of current projects. A firm with higher liquid balances is more willing to pass up a project today because it faces a lower risk of being constrained next period, and not being able to invest at all. We show that in a two-period setting, there is a critical level of liquid balances such that the firm's hurdle rate for projects initially decreases in its liquid balances up to this level, then increases, before decreasing again. In a more general multi-period setting, every period, the sensitivity of the hurdle rate to changes in liquidity decreases discontinuously as some critical level of liquid balances is reached. For some special cases, the hurdle rate is non-monotonic every period in the level of liquid balances. These results are consistent with a variety of empirical evidence. Small firms in Japan, that are likely to be financially constrained, have steadily increased the proportion of liquid balances to assets in their balance sheets since the mid-nineties; yet, investment has not picked up. Studies of the relationship

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16 We continue to assume that if the entrepreneur borrows, lenders can claim any cash carried over in the event of default. This prevents E from borrowing to carry over cash into the second period, rather than investing. See our discussion of this issue in footnote 5.
between firms' cash flows and investment have found that the cash flow sensitivity of investment is non-monotonic in the level of liquidity of firms (Kaplan and Zingales (1997), Cleary (1999, 2002)). Our results also suggest that monetary policy may not be very effective in stimulating aggregate demand through the balance sheet channel if firms have already accumulated cash balances. Recent empirical evidence showing that financially constrained firms save a bigger fraction of their cash balances in recessions (Almeida, Campello and Weisbach, 2002) provides some plausibility to this possibility.
REFERENCES


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APPENDIX

Proof of Lemma 3.

[(i)] Consider $c < c^0$ and assume that the IC constraint does not bind at the marginal state $a^e$. By Lemma 2, at $a^e$, since $E$ invests, the IR constraint does not bind, and thus by continuity, there exists states slightly lower than $a^e$ where $E$ will be able to invest and will prefer to invest rather than not invest. This contradicts the definition of $a^e$. Thus, IC must bind for all $c < c^0$ at the marginal state $a^e$. Note that we have already shown in Lemma 2 that the IR constraint does not bind.

[(ii)] Consider $c > c^0$ and assume that IR constraint does not bind at the marginal state. Now if the IC constraint also does not bind at $a^e$, by an argument analogous to the earlier part, we can show that $E$ is better off investing in a state slightly lower than $a^e$. Therefore, the IC constraint must bind at $a^e$ if IR does not bind. This means that $p[p(a^e - D) - P(0)] = B$. Further, since IR does not bind, we have $p[p(a^e - D) - P(0)] + P(0) > P(c)$. We then have $B + P(0) = P(c^e) > P(c)$. Since $c > c^0$, $P(c) > P(c^0)$ and we have a contradiction.

Since the IR constraint binds, we have $P(c) = p[p(a^e - D) - P(0)] + P(0)$. If IC also binds, we would have $p[p(a^e - D) - P(0)] = B$, so that $P(c) = B + P(0) = P(c^0)$. Since $c > c^0$, we have a contradiction.

Proof of Proposition 3.

For $c < c^0$, in the marginal state $a^e$, the IR constraint is not binding but the IC constraint is. It follows from differentiating the left hand side of the binding IC constraint that $\frac{da^e}{dc} = -\frac{1}{p}$.

On the other hand, for $c > c^0$, IR constraint must bind at the marginal state. Thus, $P(c) = pP(a^e - \frac{1-e}{p}) + (1 - p)P(0)$. Totally differentiating, we have

$$P'(c) = pP'(a^e - \frac{1-e}{p})[1/p + \frac{da^e}{dc}].$$

Simplifying, one obtains

$$\frac{da^e}{dc} = -\frac{1}{p} = \frac{P'(c)}{pP'(a^e - \frac{1-e}{p})}.$$
Hence, the result follows.

Proof of Lemma 4.

Since \( c \geq c^0 \), IR constraint must bind at \( \alpha^c \). Thus

\[
P(c) = pP(\alpha^c - (1 - c)/p) + (1 - p)P(0).
\]

(15)

If \( \bar{c} \geq c^0 \), then the IR constraint corresponding to \( \bar{c} \) also binds at its marginal state \( \alpha^{\bar{c}} \). Thus, we have

\[
P(\bar{c}) = pP(\alpha^{\bar{c}} - (1 - \bar{c})/p) + (1 - p)P(0).
\]

Since \( c > \bar{c} \), we have \( P(c) > P(\bar{c}) \) and thus \( \alpha^c - (1 - c)/p > \alpha^{\bar{c}} - (1 - \bar{c})/p \geq B \). Thus, \( c \in C \). If however \( \bar{c} < c^0 \), we know for \( \bar{c} \), the IC constraint will bind at the marginal state \( \alpha^{\bar{c}} \) and thus

\[
pP(\alpha^{\bar{c}} - (1 - \bar{c})/p) + (1 - p)P(0) \leq B + P(0) = P(c^0) < P(c).
\]

Hence from equation (15), \( \alpha^c - (1 - c)/p > \alpha^{\bar{c}} - (1 - \bar{c})/p \geq B \), and the result follows.

Proof of Proposition 6

Step 1. For any \( M, Y(M) - D(M) \leq P(0) \).

This is obvious because of our restriction that for any contract \( M \), \( D(a) \geq 0 \), thus, the set of states in which E can invest productively if the period one project fails is a subset of the states in which E can invest productively when \( D(a) = 0 \) for all \( a \). \( P(0) \) is the payoff to E when his debt obligation (to period 1 debt holders) is identically equal to zero when the period one project fails and the period two project is undertaken.

Step 2. By Lemma 4, if \( c > c^* \) and E invests in \( a \) in period 1 with the simple contract, then conditional on success, E invests in all productive states in period 2. Thus the payoff to E from investing in state \( a_1 \) using the simple debt contract is given by

\[
G = p[a_1 - (1 - c)/p + \int_{a}^{\alpha}(pa - 1)dF(a)] + (1 - p)P(0).
\]
Consider now any contract \( M = (D^a, D(a)) \) with \( D(a) \neq 0 \) for some \( a \). The payoff to \( E \) from such a contract is given by

\[
H = pP(a_1 - D^a) + (1 - p)Y(M) = p(a_1 - D^a) + \int_{a_0}^{a}(pa - 1)dF(a) + (1 - p)Y(M)
\]

where \( a^0 \geq a^* \). Since \( pD^a + (1 - p)D(M) = 1 - c \), we thus have

\[
H = p(a_1 + \int_{a_0}^{a}(pa - 1)dF(a)) - (1 - c) + (1 - p)[Y(M) + D(M)]
\]

Since \( P(0) \geq Y(M) + D(M) \), \( a^c \geq a^* \) and \( pa^* - 1 = 0 \), it follows immediately that \( G \geq H \).

Proof of Lemma 5:

**Proof.** The Lemma is clearly true for \( t = T \) since \( V^T(x) = P(x) = x + \int_{a_0}^{a}(pa - 1)dF(a) \). Assume now as an induction hypothesis that the lemma is true for all \( t = T, T - 1, \ldots, (T - m + 1) \) and let \( t = T - m \). Fix \( x > y \). There are several cases to consider depending on the exact values of \( x \) and \( y \). We will prove the result for \( 1 > x > y \). The other cases can be proved using similar arguments.

Let \( A(y) \) be the set of states in which \( E \) invests with cash holding \( y \). Let \( Q = \int_a dF(a) \) where \( a \notin A(y) \). Thus,

\[
V^t(y) = \int_{A(y)} [pV^{t+1}(a - (1 - y)/p) + (1 - p)V^{t+1}(0)]dF(a) + QV^{t+1}(y)
\]

With cash holding \( x \), \( E \) can always invest in the states \( A(y) \). Thus,

\[
V^t(x) \geq \int_{A(y)} [pV^{t+1}(a - (1 - x)/p) + (1 - p)V^{t+1}(0)]dF(a) + QV^{t+1}(x)
\]

Hence,

\[
V^t(x) - V^t(y) \geq \int_{A(y)} [pV^{t+1}(a - (1 - x)/p) - pV^{t+1}(a - (1 - y)/p)]dF(a) + Q(V^{t+1}(x) - V^{t+1}(y))
\]

By the induction hypothesis, \( [V^{t+1}(a - (1 - x)/p) - V^{t+1}(a - (1 - y)/p)] \geq (x - y)/p \) and \( V^{t+1}(x) - V^{t+1}(y) \geq (x - y) \) with strict inequality if \( y < B \). Using this, we get

\[
V^t(x) - V^t(y) \geq \int_{A(y)} [p(x - y)/p]dF(a) + Q(x - y) = (x - y)
\]
with strict inequality if $y < B$ and the Lemma is proved.

**Proof of Proposition 8:**

(i) Proof of non-monotonicity for Example 1.

For the proof, we shall need to use Lemma 5 and Lemma 6, and it can be checked easily that neither of these Lemmas are affected by the assumption that conditional on success, the firm is sold immediately to a financially unconstrained buyer. Conditional on investing at $t$ and the investment resulting in success, we now define, for $t < 1, V^{t+1}(a - \frac{(1-x)}{p}) = a - \frac{(1-x)}{p} + K(t)$, where $K(t) = \sum_{t+1}^{T} a_i \{p(a - \frac{x}{p}) - 1\}dF$ is the expected net present value of all future projects from $t + 1$ to $T$.

The proof now involves the following steps.

**Step 1.**

Claim: For any $t \leq T$, $\frac{dV^t(x)}{dx} = 1$ at $x = B$.

Proof: From Lemma 6, we know that for any $t < T$, the IR constraint must bind at $x = B$. For any $x < 1$ such that the IR constraint binds in the marginal state at $t$, we have

\[
V^t(x) = \int_{a_x^t} \left\{ pV^{t+1}(a - \frac{1-x}{p}) + (1-p)V^{t+1}(0) \right\} dF + F(a_x^t)V^{t+1}(x)
\]

\[
= \int_{a_x^t} \left\{ p(a - \frac{1-x}{p} + K(t)) + (1-p)V^{t+1}(0) \right\} dF
\]

\[
+ F(a_x^t)V^{t+1}(x).
\]

Note that $a_x^t$ denotes the marginal state, and satisfies

\[
p(a_x^t - \frac{1-x}{p} + K(t)) + (1-p)V^{t+1}(0) = V^{t+1}(x).
\]

Differentiating (16) and using (17), we get:

\[
\frac{dV^t(x)}{dx} = \int_{a_x^t} (1)dF + F(a_x^t)\frac{dV^{t+1}(x)}{dx}.
\]

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Thus,
\[
\frac{dV^t(x)}{dx}|_{x=B} = \int_{a^t_x}^B (1)dF + F(a^t_x)\frac{dV^{t+1}(x)}{dx}|_{x=B}. \tag{19}
\]

Thus, if the claim is true for \( t + 1 \), it must be true for \( t \).

We have, for \( x \leq B \)
\[
V^T(x) = x + \int_{a_x}^B (pa - 1)dF \tag{20}
\]

where
\[
a_x = \frac{B}{p} + \frac{1-x}{p}. \tag{21}
\]

For \( x > B \),
\[
V^T(x) = x + \int_{B}^x (pa - 1)dF.
\]

It is easy to show that \( V^T(x) \) is differentiable at \( x = B \), and \( \frac{dV^T(x)}{dx} = 1 \) for \( x = B \), from which the claim follows.

Step 2.

From Lemma 5, we can show that there exists \( \delta > 0 \) such that for \( x \in (B - \delta, B] \), \( V^t(x) \) is strictly concave for \( t \leq T \).

Proof: From Lemma 5, \( \frac{dV^t(x)}{dx} \geq 1 \). However, for \( x = B \), \( \frac{dV^t(x)}{dx} = 1 \). Thus, there exists \( \delta_1 \) sufficiently small such that \( \frac{dV^t(x)}{dx} \leq 0 \) over \( (B - \delta_1, B] \).

Suppose \( \frac{dV^t(x)}{dx} = 0 \) over \( (B - \delta_1, B] \). Then \( \frac{dV^t(x)}{dx} = k \) over this interval.

We cannot have \( k > 1 \), since \( \frac{dV^t(x)}{dx} \to 1 \) as \( x \to B \). Thus, \( k = 1 \), and \( V^t(B) = V^t(B - \delta_1) = \delta_1 \), contradicting the Lemma (which says that a strict inequality must obtain).

Step 3.

Now we know that the IR constraint must bind for \( c > c' \). Differentiating the IR constraint for \( c \in (B - \ell, B) \) and using concavity, we get non-monotonicity.
(ii) Proof of non-monotonicity for Example 2.

Once again, Lemmas 5 and 6 can be shown to hold for this example. Notice that, if at time $t$, $E$ has $c > 1$, then we can now assume that $E$ consumes $c - 1$ and keeps 1. This is because, if $E$ invests 1 and the outcome is a success, he will receive more than 1, so that he will be again able to invest next period. On the other hand, if the outcome is a failure, then by assumption, $E$ does not invest any more. His options to invest are exactly the same if he retained any amount greater than 1.

This implies that we can write, for $c > 1$ and $t = 1, 2, \ldots, T$

$$V^t(c) = c - 1 + V^{t+1}(1)$$

where $V^t(1)$ is defined as follows for $t = 1, 2, \ldots, T - 1$:

$$V^t(1) = \int_{a^t_1} p(a - 1 + V^{t+1}(1))dF + F(a^t_1)V^{t+1}(1)$$

and $a^t_1$ maximizes the expression on the right hand side. The first-order condition gives

$$(1 - p)V^{t+1}(1) = p(a^t_1 - 1).$$

For $t = T$, $V^t(1)$ is defined as

$$V^T(1) = \int_{a^T} padF + F(a^T_1).1,$$

where $a^T_1$ maximizes the right hand side expression, and is equal to $a^* = 1/p$. Thus

$$V^T(1) = \int_{a^*} (pa - 1)dF + 1.$$

The rest of the proof follows steps very similar to those for Example 1. Except for step 1, the other steps are almost identical. In step 1, we now claim:

Claim: For any $t \leq T$, $\frac{dV^t(x)}{dx} = 1$ for $1 \geq x \geq B$. 

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Proof: Note that we already know from equation (22) that for \( x > 1 \),
\[
\frac{dV^t(x)}{dx} = 1.
\]
It can be shown that for any \( t \leq T \), the IR constraint must bind
at \( x \geq B \) in the marginal state. For any \( x \) such that the IR constraint binds
in the marginal state at \( t \), we have

\[
V^t(x) = \int_{a_x^t} \left\{ pV^{t+1}(a - \frac{1-x}{p}) \right\} dF + F(a_x^t)V^{t+1}(x)
\]
(23)

where \( a_x^t \) denotes the marginal state, and satisfies

\[
pV^{t+1}(a_x^t - \frac{1-x}{p}) = V^{t+1}(x).
\]
(24)

Differentiating (23) and using (24), we get:

\[
\frac{dV^t(x)}{dx} = \int_{a_x^t} \left\{ p \frac{dV^{t+1}(a - \frac{1-x}{p})}{dx} \right\} dF + F(a_x^t) \frac{dV^{t+1}(x)}{dx}
\]
(25)

From (24), it is easily checked that \( a_x^t - \frac{1-x}{p} > x \). Thus, for \( x \geq B \) and
\( a \geq a_x^t \), we have \( a - \frac{(1-x)}{p} > x \geq B \). If the claim to be proved is
true for \( t + 1 \), and we have \( 1 > a - \frac{1-x}{p} > x \geq B \), then
\[
\frac{dV^{t+1}(a - \frac{(1-x)}{p})}{dx} = p \frac{dV^{t+1}(a - \frac{(1-x)}{p})}{dx} (1/p) = 1.
\]
If \( a - \frac{1-x}{p} > 1 \), we already know that \( \frac{dV^{t+1}(a - \frac{(1-x)}{p})}{dx} =
1 \), so that again, \( \frac{dV^{t+1}(a - \frac{(1-x)}{p})}{dx} = 1 \). Thus, from equation (25), if the claim
is true for \( t + 1 \), it must be true for \( t \). To complete the step, it can be easily
shown that the claim is true for \( t = T \).
Figure 1: Single-period payoff $P^1$ as a function of current cash balance $x$ for the case $P^1(0) = 0$: $p = 1/2$, $\bar{a} = 3$, $b = 3/4$. 
Figure 2: Single-period payoff $P^2$ as a function of current cash balance $x$ for the case $P^1(0) > 0$: $p = 1/2$, $a = 4$, $b = 3/4$. 
Figure 3: Non-monotonicity of hurdle rates as functions of the initial cash balance $c$: Solid line, $A_1^1$; dashed line, $A_2^2$; short-dashed line, $A_3^2$; dotted line, $A_4^4$. $p = 3/4$, $a = 2$, $b = 3/4$. 