Some progress in non-Archimedean analysis

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ABSTRACT. In this paper, we will introduce some results and problems obtained lately that relate to unique range sets, the *abc*-conjecture and the Shiffman conjecture for non-Archimedean meromorphic functions.

1. Introduction

Mason (see [19],[20],[18]), Stothers [28] and others started one recent trend of thoughts by discovering an entirely new relation among polynomials as follows. Let $f(z)$ be a polynomial with coefficients in an algebraically closed field κ of characteristic 0 and let $\bar{n}\left(\infty, \frac{1}{f}\right)$ be the number of distinct zeros of f .

THEOREM 1.1. *Let $a(z)$, $b(z)$, $c(z)$ be relatively prime polynomials in an algebraically closed field κ of characteristic 0 and not all constant such that $a + b = c$. Then*

$$\max\{\deg(a), \deg(b), \deg(c)\} \leq \bar{n}\left(\infty, \frac{1}{abc}\right) - 1.$$

Let a be a non-zero integer. Define the radical of a to be

$$r(a) = \prod_{p|a} p$$

i.e., the product of the distinct prime factors of a . There is a classical analogy between polynomials and integers. Under that analogy, $\bar{n}\left(\infty, \frac{1}{f}\right)$ of a polynomial f corresponds to $\log r(a)$ of an integer a . Influenced by Theorem 1.1 which is known in some of the literature as Mason's theorem (cf. [17]), and considerations of Szpiro, Frey, Masser and Oesterlé formulated the *abc*-conjecture for integers as follows:

CONJECTURE 1.2. *Given $\varepsilon > 0$, there exists a number $C(\varepsilon)$ having the following property. For any nonzero relatively prime integers a , b , c such that $a + b = c$,*

$$\max\{|a|, |b|, |c|\} \leq C(\varepsilon)r(abc)^{1+\varepsilon}.$$

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Further, by comparing Nevanlinna theory in complex analysis with Roth's theorem in number theory, P. Vojta [30] published his famous dictionary. Based on it, he formulated the Griffiths-Lang conjecture in value distribution theory as a problem in number theory, which now is called Vojta's conjecture. Moreover, the *abc* conjecture is a consequence of the Vojta Conjecture (see Vojta [30], [32]). In this paper, we will introduce some progresses in this direction, some new and generalized results and related conjectures of Vojta [31], and also report some studies in the uniqueness of meromorphic functions. We will focus our efforts mainly on non-Archimedean cases.

2. Notes on the *abc*-conjecture

Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Let K denote the field κ or $\mathbb{C}$ and set $K_* = K \setminus \{0\}$. Let $\mathcal{M}(K)$ (resp., $\mathcal{A}(K)$) be the set of meromorphic (resp., entire) functions in K . Take $f \in \mathcal{M}(K)$ and take $a \in K \cup \{\infty\}$. Let $\mu_f^a(z_0)$ denote the *a-valued multiplicity* of f at z_0 , that is, $\mu_f^a(z_0) = m$ if and only if

$$f(z) = \begin{cases} a + (z - z_0)^m h(z) & : a \neq \infty \\ \frac{h(z)}{(z - z_0)^m} & : a = \infty \end{cases}$$

with $h(z_0) \neq 0, \infty$. By the definition, we obtain a function

$$\mu_f^a : K \longrightarrow \mathbb{Z}_+$$

with $\mu_f^a(z_0) > 0$ for some $z_0 \in K$ if and only if $f(z_0) = a$. For a positive integer k , define a function

$$\mu_{f,k}^a(z) = \begin{cases} \mu_f^a(z) & : \mu_f^a(z) \leq k \\ k & : \mu_f^a(z) > k. \end{cases}$$

Further, for a fixed real number $\rho_0 > 0$ write

$$n_k \left(r, \frac{1}{f-a} \right) = \sum_{|z| \leq r} \mu_{f,k}^a(z),$$

and

$$N_k \left(r, \frac{1}{f-a} \right) = \int_{\rho_0}^r n_k \left(t, \frac{1}{f-a} \right) \frac{dt}{t}.$$

Given a non-constant entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (a_n \in \kappa)$$

on κ . Define the *maximum term*:

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n.$$

Let $n \left(r, \frac{1}{f} \right)$ denote the number of zeros (counting multiplicity) of f with absolute value $\leq r$, that is,

$$n \left(r, \frac{1}{f} \right) = \sum_{|z| \leq r} \mu_f^0(z).$$

Define the *valence function* of f for 0 by

$$N\left(r, \frac{1}{f}\right) = \int_{\rho_0}^r \frac{n\left(t, \frac{1}{f}\right)}{t} dt \quad (r > \rho_0).$$

We also denote the number of distinct zeros of f on $\{|z| \leq r\}$ by $\bar{n}\left(r, \frac{1}{f}\right)$, equivalently,

$$\bar{n}\left(r, \frac{1}{f}\right) = n_1\left(r, \frac{1}{f}\right) = \sum_{|z| \leq r} \mu_{f,1}^0(z),$$

and define

$$\bar{N}\left(r, \frac{1}{f}\right) = \int_{\rho_0}^r \frac{\bar{n}\left(t, \frac{1}{f}\right)}{t} dt \quad (r > \rho_0).$$

By a *meromorphic function* f on κ we will mean the quotient $\frac{g}{h}$ of two entire functions g and h such that g and h do not have any common factors in the ring of entire functions on κ . We can uniquely extend μ to meromorphic function $f = \frac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)}.$$

Define the *counting function* $n(r, f)$ and the *valence function* $N(r, f)$ of f for poles respectively by

$$n(r, f) = n\left(r, \frac{1}{h}\right) \quad \text{and} \quad N(r, f) = N\left(r, \frac{1}{h}\right).$$

Then one has the *Jensen formula*:

$$(2.1) \quad N\left(r, \frac{1}{f}\right) - N(r, f) = \log \mu(r, f) - \log \mu(\rho_0, f).$$

Define

$$m(r, f) = \log^+ \mu(r, f) = \max\{0, \log \mu(r, f)\}.$$

Finally, we define the *characteristic function*:

$$T(r, f) = m(r, f) + N(r, f).$$

Then the following formula

$$(2.2) \quad T(r, f) = \max\left\{N\left(r, \frac{1}{f-a}\right), N\left(r, \frac{1}{f-b}\right)\right\} + O(1)$$

holds for any two distinct elements $a, b \in \kappa \cup \{\infty\}$ (cf. Hu-Yang [11], [10]). By using the second main theorem and the formula (2.2), we obtained

THEOREM 2.1 (cf. Hu-Yang [11], [10]). *Let $a(z)$, $b(z)$ and $c(z)$ be entire functions in κ without common zeros and not all constants such that $a + b = c$. Then*

$$\max\{T(r, a), T(r, b), T(r, c)\} \leq \bar{N}\left(r, \frac{1}{abc}\right) - \log r + O(1).$$

The following fact can be easily deduced by the second main theorem due to Nevanlinna.

THEOREM 2.2 (cf. [29]). Let $a(z)$, $b(z)$ and $c(z)$ be entire functions of order $\leq \rho < +\infty$ in $\mathbb{C}$ without common zeros and not all constants such that $a + b = c$. Then for any $\varepsilon > 0$, one has

$$\max \left\{ T \left(r, \frac{a}{c} \right), T \left(r, \frac{b}{c} \right) \right\} \leq \overline{N} \left(r, \frac{1}{abc} \right) + (\rho + \varepsilon - 1) \log r + O(1).$$

Above two theorems are direct generalizations of Theorem 1.1. In Hu-Yang [14], we further generalized Theorem 2.1 and Theorem 1.1 as follows.

THEOREM 2.3. Let $f_j (j = 0, \dots, k (\geq 2))$ be entire functions on κ such that $f_1, \dots, f_k$ are linearly independent on κ . Let

$$(2.3) \quad f_1 + \dots + f_k = f_0,$$

and assume that f_j and f_0 have no common zeros for $j = 1, \dots, k$. Then

$$\max_{0 \leq j \leq k} \{T(r, f_j)\} \leq \sum_{i=0}^k N_{k-1} \left(r, \frac{1}{f_i} \right) - \frac{k(k-1)}{2} \log r + O(1).$$

Theorem 2.3 implies directly the following fact:

COROLLARY 2.4. Let $f_j (j = 0, \dots, k (\geq 2))$ be polynomials on κ such that $f_1, \dots, f_k$ are linearly independent on κ . Let

$$(2.4) \quad f_1 + \dots + f_k = f_0,$$

and assume that f_j and f_0 are coprime for $j = 1, \dots, k$. Then

$$\max_{0 \leq j \leq k} \{\deg(f_j)\} \leq \sum_{i=0}^k n_{k-1} \left(\infty, \frac{1}{f_i} \right) - \frac{k(k-1)}{2},$$

where

$$n_{k-1} \left(\infty, \frac{1}{f_i} \right) = \lim_{r \rightarrow \infty} n_{k-1} \left(r, \frac{1}{f_i} \right).$$

The proof of Theorem 2.3 can be completed by using the formula (2.2) and the following more general result which is a non-Archimedean analogue of a result of Nevanlinna[21]:

LEMMA 2.5 (cf. Hu-Yang [10], [14]). Let $f_j (j = 1, \dots, k)$ be linearly independent meromorphic functions on κ such that

$$(2.5) \quad f_1 + \dots + f_k = 1.$$

Then

$$(2.6) \quad \begin{aligned} T(r, f_j) &< \sum_{i=1}^k N \left(r, \frac{1}{f_i} \right) - \sum_{i \neq j} N(r, f_i) + N(r, \mathbf{W}) \\ &\quad - N \left(r, \frac{1}{\mathbf{W}} \right) - \frac{k(k-1)}{2} \log r + O(1), \quad 1 \leq j \leq k \end{aligned}$$

where $\mathbf{W}$ is the Wronskian of $f_1, \dots, f_k$.

Let a be a non-zero integer. Then a can be expressed as

$$|a| = p_1^{i_1} \cdots p_n^{i_n},$$

for distinct primes $p_1, \dots, p_n$. Define

$$r_k(a) = \prod_{\nu=1}^n p_{\nu}^{\min\{i_{\nu}, k\}}.$$

Then an analogue of Corollary 2.4 for integers would be the following problem:

CONJECTURE 2.6 (cf. Hu-Yang [14]). *Let $a_j (j = 0, \dots, k)$ be nonzero integers such that a_j, a_0 are coprime for $j = 1, \dots, k$,*

$$(2.7) \quad a_1 + \dots + a_k = a_0,$$

and no proper subsum of (2.7) is equal to 0. Then for $\varepsilon > 0$, there exists a number $C(\varepsilon, k)$ such that

$$\max_{0 \leq j \leq k} \{|a_j|\} \leq C(\varepsilon, k) \left\{ \prod_{i=0}^k r_{k-1}(a_i) \right\}^{1+\varepsilon}.$$

We also think that Conjecture 2.6 can be generalized as follows:

CONJECTURE 2.7 ([12]). *Let $a_j (j = 0, \dots, k)$ be nonzero integers with $k \geq 2$ such that the greatest common divisor of $a_0, \dots, a_k$ is 1,*

$$(2.8) \quad a_1 + \dots + a_k = a_0,$$

and no proper subsum of (2.8) is equal to 0. Then for $\varepsilon > 0$, there exists a number $C(\varepsilon, k)$ such that

$$\max_{0 \leq j \leq k} \{|a_j|\} \leq C(\varepsilon, k) \left\{ \prod_{i=0}^k r_{k-1}(a_i) \right\}^{1+\varepsilon}.$$

Relating the above conjecture, we were reminded by W. Cherry [3] that the following conjecture was proposed by Vojta.

CONJECTURE 2.8 ([31]). *If $a_0, \dots, a_k$ are nonzero integers such that*

$$a_1 + \dots + a_k = a_0$$

and the greatest common divisor of $a_0, \dots, a_k$ is 1, then

$$\max\{|a_0|, \dots, |a_k|\} \leq Cr(a_0 \dots a_k)^{1+\varepsilon}$$

for all $a_0, \dots, a_k$ as above outside a proper Zariski-closed subset of the hyperplane $x_1 + \dots + x_k = x_0$ in $\mathbb{P}^k$.

Furthermore, it was indicated by Vojta in [31] that the above conjecture would be derived from his main conjecture.

CONJECTURE 2.9 ([16]). *Take positive integers $n, l_j (1 \leq j \leq n)$ such that*

$$n \geq 2, \quad \alpha = 1 - \sum_{j=1}^n \frac{n-1}{l_j} > 0.$$

Let $a_j (j = 1, \dots, n)$ be nonzero integers such that $\gcd(a_1, \dots, a_n) = 1$. If there exist $\delta_j \in \{-1, 1\} (j = 1, \dots, n)$ such that

$$(2.9) \quad \delta_1 a_1^{l_1} + \dots + \delta_n a_n^{l_n} \neq 0,$$

and no proper subsum of (2.9) is equal to 0, then for $\varepsilon > 0$, there exists a number $C = C(n, \varepsilon)$ depending on n and ε such that

$$(2.10) \quad \max_{1 \leq j \leq n} \{|a_j|^{\alpha_{l_j}}\} \leq Cr_{n-1} \left(\sum_{j=1}^n \delta_j a_j^{l_j} \right)^{1+\varepsilon}.$$

Conjecture 2.9 is a generalization of the weaker version of Hall's conjecture, and follows easily from Conjecture 2.7. Also some analogue of Conjecture 2.9 for polynomials has been proved in [16]. In particular, for the case

$$(2.11) \quad n = 2, \quad l_1 = 2, \quad l_2 = 3,$$

Hall's conjecture (cf. [26]) states that there exists a constant C such that

$$(2.12) \quad |y|^{\frac{1}{2}} < C(\varepsilon) |x^2 + y^3|^{1+\varepsilon}$$

holds for integers x, y with $x^2 - y^3 \neq 0$. However, Conjecture 2.9 claims that given $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$(2.13) \quad \max \left\{ |x|^{\frac{1}{3}}, |y|^{\frac{1}{2}} \right\} \leq C(\varepsilon) r_1 (x^2 + y^3)^{1+\varepsilon}$$

holds for nonzero integers x, y with $x^2 + y^3 \neq 0$, $\gcd(x, y) = 1$. Obviously, the inequality is stronger than (2.12) under above stronger conditions. The analogue of Conjecture 2.9 for polynomials is proved in [16].

3. Unique range sets of meromorphic functions

Take $f \in \mathcal{M}(K)$ and take $a \in K \cup \{\infty\}$. For a positive integer k , define a set

$$E_f(a, k) = \{(\mu_{f,k}^a(z), z) \mid z \in K\}.$$

Take a non-empty set $S \subset K \cup \{\infty\}$ and write

$$E_f(S, k) = \bigcup_{a \in S} \{(\mu_{f,k}^a(z), z) \mid z \in K\} = \bigcup_{a \in S} E_f(a, k).$$

Usually, we write

$$E_f(S) = E_f(S, \infty), \quad \overline{E}_f(S) = E_f(S, 1).$$

Given a family $\mathcal{F}$ of $\mathcal{M}(K)$, a non-empty set S in $K \cup \{\infty\}$ is called a *urscm* (resp., *ursim*) for $\mathcal{F}$ if for any non-constant functions $f, g \in \mathcal{F}$ satisfying $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$), one has $f = g$, where “urscm” (resp., “ursim”) means a *unique range set for counting multiplicity* (resp., *unique range set for ignoring multiplicity*). If two functions $f, g \in \mathcal{F}$ satisfy $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$), we also say that f and g share the set S CM (resp., IM). These notations were introduced by Gross-Yang [8].

In the research of unique range sets of meromorphic functions on $\mathbb{C}$, the following type of polynomials have been used by several researchers (cf. [34])

$$(3.1) \quad Y_{n,m}(z) = Y_{n,m}(a, b; z) = z^n - az^m + b,$$

where $n, m \in \mathbb{Z}^+$, $n > m$, and where we will let $a, b \in K_*$ with

$$(3.2) \quad \frac{a^n}{b^{n-m}} \neq \frac{n^n}{m^m(n-m)^{n-m}}.$$

The condition (3.2) makes $Y_{n,m}$ only has simple zeros. We will denote the set of zeros of $Y_{n,m}$ by $\dot{Y}_{n,m}$. We also consider the following polynomial

$$(3.3) \quad F_{n,b}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + b,$$

where $b \in K \setminus \{0, -1\}$, and denote the set of zeros of $F_{n,b}$ by $\dot{F}_{n,b}$. Note that $\dot{F}_{n,b}$ contains n distinct elements. G. Frank and M. Reinders [6] proved that the set $\dot{F}_{n,b}$ is a urscm for $\mathcal{M}(\mathbb{C})$ if $n \geq 11$. In Hu-Yang [13], we obtained

THEOREM 3.1. *Take an integer $n \geq 3$. If the constants $a, b \in \kappa_*$ in the polynomial $Y_{n,n-1}$ also satisfy*

$$(3.4) \quad \frac{a^n}{b} \neq 2 \frac{n^n}{(n-1)^{n-1}},$$

then the set $\dot{Y}_{n,n-1}$ is a urscm for $\mathcal{A}(\kappa)$.

If $n \geq 4$, Theorem 3.1 is proved respectively by Boutabaa-Escassut-Haddad [1], and Hu-Yang [9], and the condition (3.4) can be canceled in this case. Escassut [5] and Cherry [3] pointed out separately to us that Theorem 3.1 can be deduced simply by using the results in [1] (for the case $n = 3$) and [4], respectively. In our proof, a key point is to use the formula (2.2).

For more general settings, H. Fujimoto [7] gave some sufficient conditions for a finite subset S of $\mathbb{C}$ to be a unique range set for meromorphic (or entire) functions which particularly implies the following fact:

THEOREM 3.2. *Let f and g be non-constant meromorphic (resp., entire) functions on $\mathbb{C}$ satisfying*

$$E_f(\dot{F}_{n,b}, k) = E_g(\dot{F}_{n,b}, k) \quad (2 \leq k \in \mathbb{Z}^+).$$

If $n > 10 + \frac{4}{k-1}$ (resp., $n > 6 + \frac{2}{k-1}$), then $f = g$.

In Hu-Yang [13], we gave a new approach of H. Fujimoto's results. In particular, we obtained

THEOREM 3.3. *Let f and g be non-constant meromorphic (resp., entire) functions on $\mathbb{C}$ satisfying*

$$E_f(\dot{F}_{n,b}, k) = E_g(\dot{F}_{n,b}, k) \quad (2 \leq k \in \mathbb{Z}^+).$$

If $n > 10 + \frac{4}{k}$ (resp., $n > 6 + \frac{2}{k}$), then $f = g$.

Obviously, Theorem 3.3 is better than Theorem 3.2 for the case $k = 2$, and is same as Theorem 3.2 for larger k . To prove Theorem 3.3, we used the Frank-Reinders' technique [6], that is, we first proved the following lemma (which is stated for the non-Archimedean case).

LEMMA 3.4 ([13], [10]). *Let F and G be non-constant meromorphic functions on κ satisfying*

$$E_F(\infty, k) = E_G(\infty, k) \quad (2 \leq k \in \mathbb{Z}^+),$$

and let $a_1, \dots, a_q$ be distinct elements of κ with $q \geq 2$. Then one of the following cases must occur

1):

$$\begin{aligned} \left(q - \frac{3}{2}\right) \{T(r, F) + T(r, G)\} &\leq \sum_{j=1}^q \left\{ N_2 \left(r, \frac{1}{F - a_j} \right) + N_2 \left(r, \frac{1}{G - a_j} \right) \right\} \\ &\quad + N_{k+1}(r, F) - N_k(r, F) + N_{k+1}(r, G) - N_k(r, G) \\ &\quad - \log r + O(1). \end{aligned}$$

2): $G = AF + B$, where $A, B \in \kappa$ with $A \neq 0$, and

$$\#(\{a_1, \dots, a_q\} \cap \{Aa_1 + B, \dots, Aa_q + B\}) \geq 2.$$

Let $\text{Aut}(K)$ be the group of non-constant linear polynomials in $K[z]$, that is, $\sigma \in \text{Aut}(K)$ if and only if $\sigma(z) = az + b$ with $a \neq 0$. Let id be the *identity mapping* in K . Note that a set S in K is an ursim for $\text{Aut}(K)$ if and only if S is an urscm for $\text{Aut}(K)$. Take $\sigma, \eta \in \text{Aut}(K)$. We see that $\sigma^{-1}(S) = \eta^{-1}(S)$ if and only if $\eta \circ \sigma^{-1}(S) = S$. Thus S is an ursim for $\text{Aut}(K)$ if and only if there exists no $\xi \in \text{Aut}(K) \setminus \{id\}$ such that $\xi(S) = S$.

EXAMPLE 3.5. Set $\Omega_n(K) = \{z \mid z^n = 1\}$ and define $\sigma, \eta \in \text{Aut}(K)$ by

$$\sigma(z) = -z, \quad \eta(z) = wz \quad (w \in \Omega_3(K) \setminus \{1\}).$$

Obviously, one has

$$\sigma(\{-1, 0, 1\}) = \{-1, 0, 1\}, \quad \eta(\Omega_3(K)) = \Omega_3(K).$$

Hence $\{-1, 0, 1\}$ and $\Omega_3(K)$ are not ursim for $\text{Aut}(K)$. Note that for any distinct elements $a, b \in K$, the mapping $\xi \in \text{Aut}(K)$ defined by

$$\xi(z) = -z + a + b$$

satisfies $\xi(\{a, b\}) = \{a, b\}$. Hence an ursim for $\text{Aut}(K)$ contains at least 3 distinct elements.

EXAMPLE 3.6. Take $S = \{a_1, a_2, a_3\} \subset K$ and set

$$c = \frac{1}{3}(a_1 + a_2 + a_3), \quad a = c - a_1, \quad w \in \Omega_3(K) \setminus \{1\}.$$

Then S is not an ursim for $\text{Aut}(K)$ if and only if it is one of the following two forms (see [1]):

- (i) $S = \{c - a, c, c + a\}$;
- (ii) $S = \{c + a, c + wa, c + w^2a\}$.

Two mappings in $\text{Aut}(K)$ under which S are invariant are respectively

$$\sigma(z) = -z + 2c, \quad \eta(z) = wz + (1 - w)c.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two families of $\mathcal{M}(K)$ such that $\mathcal{E} \subset \mathcal{F}$. By the definition, if a non-empty subset S in $K \cup \{\infty\}$ is a urscm (resp., ursim) for $\mathcal{F}$, then S also is a urscm (resp., ursim) for $\mathcal{E}$. A natural question is what we can say about the converse. In this direction, Boutabaa, Escassut and Haddad [1] asked whether a finite set S in κ is an urscm for $\mathcal{A}(\kappa)$ if and only if S is an ursim for $\text{Aut}(\kappa)$ (Remark: They used different notation). They confirmed this case if either S has only three different points or $\mathcal{A}(\kappa)$ is replaced by $\kappa[x]$. The general case is proved by Cherry and Yang [4]:

THEOREM 3.7. A finite set S in κ is an urscm for $\mathcal{A}(\kappa)$ if and only if S is an ursim for $\text{Aut}(\kappa)$.

Hu and Yang [15] prove the analogue of Theorem 3.7 for entire functions of order < 1 on $\mathbb{C}^m$. Applying the result, we can prove the following fact:

THEOREM 3.8 ([10]). *For $b \in \kappa_* \setminus \{-1, -2\}$, the set $\dot{F}_{3,b}$ is an urscm for $\mathcal{A}(\kappa)$.*

Related to Theorem 3.7, the following problems are open:

CONJECTURE 3.9 ([4]). *A finite set S in κ is an ursim for $\mathcal{A}(\kappa)$ if and only if S is an ursim for $\kappa[z]$.*

CONJECTURE 3.10 (cf. [10]). *A finite set S in κ is an urscm (resp., ursim) for $\mathcal{M}(\kappa)$ if and only if S is an urscm (resp., ursim) for $\kappa(z)$.*

C. C. Yang [33] conjectured that for any two distinct values $a, b \in \mathbb{C}$, if two non-constant polynomials f and g of same degree on $\mathbb{C}$ share $S = \{a, b\}$ IM, then $f = g$ or $f + g = a + b$. This conjecture and its generalization were proved by F. Pakovitch [23] and by I. V. Ostrovskii, F. Pakovitch and M. G. Zaidenberg [22], respectively. We [10] conjectured that if f and g are two non-constant transcendental entire functions (or polynomials of the same degree) on κ sharing the set $S = \{a, b\} \subset \kappa$ of two distinct values a and b IM, then $f = g$ or $f + g = a + b$. However, this conjecture is not true if f, g are transcendental. Recently, W. Cherry [3] showed, by using polynomials

$$P(z) = 2z^2(2z - 3) + 1, \quad Q(z) = -\frac{8}{3}z(z - 1) + 1,$$

which share $\{-1, 1\}$ IM, and an arbitrary transcendental entire function h on κ , that transcendental entire functions $f = P(h)$ and $g = Q(h)$ on κ share $\{-1, 1\}$ IM, but clearly $f \neq g$ and $f + g \neq 0$. Furthermore, we note here that f, g satisfy

$$\lim_{r \rightarrow \infty} \frac{T(r, g)}{T(r, f)} = \frac{2}{3}.$$

Thus we have the following modified question:

CONJECTURE 3.11. *For any two distinct values $a, b \in \kappa$, if two non-constant entire functions f and g on κ share $S = \{a, b\}$ IM and satisfy*

$$\lim_{r \rightarrow \infty} \frac{T(r, g)}{T(r, f)} = 1,$$

then $f = g$ or $f + g = a + b$.

If f and g share $S = \{a, b\}$ CM, it is easily reduced from Theorem 2 in [4] (see Hu-Yang [10]).

4. Notes on the Shiffman conjecture

Let V be a normed vector space of dimension $n + 1 > 0$ over κ and also let $|| \cdot ||$ denote a norm defined over a base $e = (e_0, \dots, e_n)$ of V . Write the projective space $\mathbb{P}(V) = V/\kappa_*$ and let $\mathbb{P} : V \setminus \{0\} \rightarrow \mathbb{P}(V)$ be the standard projection. If $S \subset V$, abbreviate

$$\mathbb{P}(S) = \mathbb{P}(S \cap V_*), \quad V_* = V - \{0\}.$$

The dual vector space V^* of V consists of all κ -linear functions $\alpha : V \rightarrow \kappa$, and we shall call

$$\langle \xi, \alpha \rangle = \alpha(\xi)$$

the *inner product* of $\xi \in V$ and $\alpha \in V^*$. If $\alpha \neq 0$, the n -dimensional linear subspace

$$E[a] = E[\alpha] = \text{Ker}(\alpha) = \alpha^{-1}(0)$$

depends on $a = \mathbb{P}(\alpha) \in \mathbb{P}(V^*)$ only, and $\ddot{E}[a] = \mathbb{P}(E[a])$ is a *hyperplane* in $\mathbb{P}(V)$. Thus $\mathbb{P}(V^*)$ bijectively parameterizes the hyperplanes in $\mathbb{P}(V)$.

The set of all symmetric vectors in $\otimes_m V$ is a linear subspace of $\otimes_m V$, denoted by $\Pi_m V$, called the *m-fold symmetric tensor product of V*. Then

$$\dim \Pi_m V = \binom{n+m}{m}.$$

For $\xi_j \in V$ ($j = 1, \dots, m$), let $\xi_1 \Pi \dots \Pi \xi_m$ be the symmetric tensor product. For $\xi \in V$, let $\xi^{\Pi m}$ be the m -th symmetric tensor power, and define

$$x^{\Pi m} = \mathbb{P}(\xi^{\Pi m})$$

for $x = \mathbb{P}(\xi)$.

By a *(non-Archimedean) holomorphic curve*

$$f : \kappa \longrightarrow \mathbb{P}(V),$$

we mean an equivalence class of $(n+1)$ -tuples of entire functions

$$(\tilde{f}_0, \dots, \tilde{f}_n) : \kappa \longrightarrow \kappa^{n+1}$$

such that $\tilde{f}_0, \dots, \tilde{f}_n$ have no common factors in the ring of entire functions on κ and such that not all of the $\tilde{f}_j$ are identically zero. Here

$$\tilde{f} = \tilde{f}_0 e_0 + \dots + \tilde{f}_n e_n : \kappa \longrightarrow V$$

will be called a *reduced representation of f*. Define

$$\mu(r, \tilde{f}) = \max_{0 \leq k \leq n} \mu(r, \tilde{f}_k).$$

Then the *characteristic function*

$$T(r, f) = \log \mu(r, \tilde{f})$$

is well defined for all $r > 0$, up to $O(1)$.

The holomorphic curve f induces a holomorphic curve

$$f^{\Pi m} : \kappa \longrightarrow \mathbb{P}(\Pi_m V)$$

with a reduced representation $\tilde{f}^{\Pi m} : \kappa \longrightarrow \Pi_m V$ defined by

$$\tilde{f}^{\Pi m}(z) = \tilde{f}(z)^{\Pi m}, \quad z \in \kappa.$$

Let $g : \kappa \longrightarrow \mathbb{P}(\Pi_m V^*)$ be a non-Archimedean holomorphic curve such that the pair $(f^{\Pi m}, g)$ is free, that is, $\langle \tilde{f}^{\Pi m}, \tilde{g} \rangle \neq 0$. Then the first main theorem implies

$$(4.1) \quad N_{f^{\Pi m}}(r, g) \leq mT(r, f) + T(r, g) + O(1),$$

where

$$N_{f^{\Pi m}}(r, g) = N\left(r, \frac{1}{\langle \tilde{f}^{\Pi m}, \tilde{g} \rangle}\right).$$

Take non-negative integers a and b with $a \leq b$. Let J_a^b be the set of all increasing injective mappings $\lambda : \mathbb{Z}[0, a] \longrightarrow \mathbb{Z}[0, b]$, where

$$\mathbb{Z}[m, n] = \{i \in \mathbb{Z} \mid m \leq i \leq n\}.$$

Take a sequence $\{m_0, m_1, \dots, m_q\}$ of positive integers. Let $\mathcal{A} = \{a_0, a_1, \dots, a_q\}$ be a family of points $a_j \in \mathbb{P}(\Pi_{m_j} V^*)$. Take $\alpha_j \in \Pi_{m_j} V^* \setminus \{0\}$ with $\mathbb{P}(\alpha_j) = a_j$, and define

$$\tilde{\alpha}_j(\xi) = \langle \xi^{\Pi_{m_j}}, \alpha_j \rangle, \quad \xi \in V, \quad j = 0, 1, \dots, q.$$

The family $\mathcal{A} = \{a_0, a_1, \dots, a_q\}$ ($q \geq n$) is said to be admissible if, for every $\lambda \in J_n^q$, the system

$$(4.2) \quad \tilde{\alpha}_{\lambda(i)}(\xi) = 0, \quad i = 0, 1, \dots, n$$

has only the trivial solution $\xi = 0$ in V .

THEOREM 4.1 (Hu-Yang [10]). *Let $\mathcal{A} = \{a_j\}_{j=0}^q$ be a finite admissible family of points $a_j \in \mathbb{P}(\Pi_{m_j} V^*)$ with $n < q$. Let $f : \kappa \rightarrow \mathbb{P}(V)$ be a holomorphic curve which is algebraically non-degenerate. Set*

$$d = \max_{\lambda \in J_n^q} \left\{ \frac{1}{n+1} \sum_{i=0}^n m_{\lambda(i)} \right\}.$$

Then

$$\begin{aligned} \left(\sum_{j=0}^q m_j - d(n+1) \right) T(r, f) &\leq \sum_{j=0}^q N_{f \Pi_{m_j}}(r, a_j) - N_{\text{Ram}}(r, f) \\ &\quad - \frac{n(n+1)}{2} \log r + O(1), \end{aligned}$$

where

$$N_{\text{Ram}}(r, f) = \min_{\lambda \in J_n^q} N \left(r, \frac{1}{\mathbf{W}_\lambda} \right)$$

in which $\mathbf{W}_\lambda$ is the Wronskian of $\tilde{\alpha}_{\lambda(0)} \circ \tilde{f}, \dots, \tilde{\alpha}_{\lambda(n)} \circ \tilde{f}$.

Theorem 4.1 is an analogue of the Shiffman conjecture in the complex variables (cf. [27]). If the ramification term is not involved in the second main theorem, Min Ru [24] observed that the first main theorem yields directly the following fact:

THEOREM 4.2. *Let $\mathcal{A} = \{a_j\}_{j=0}^q$ be a finite admissible family of points $a_j \in \mathbb{P}(\Pi_m V^*)$ with $n < q$. Let $f : \kappa \rightarrow \mathbb{P}(V)$ be a holomorphic curve satisfying $f(\kappa) \not\subset \tilde{E}[a_j]$ for $j = 0, \dots, q$. Then*

$$(q+1-n) m T(r, f) \leq \sum_{j=0}^q N_{f \Pi_m}(r, a_j) + O(1).$$

In fact, under the conditions of Theorem 4.2, we have the following stronger formula (cf. [10]):

$$(4.3) \quad m T(r, f) = \max_{0 \leq i \leq n} N_{f \Pi_m}(r, a_{\lambda(i)}) + O(1), \quad \lambda \in J_n^q.$$

Theorem 4.2 can then be deduced immediately by (4.3). Without loss of generality, we may assume that

$$N_{f \Pi_m}(r, a_0) \leq N_{f \Pi_m}(r, a_1) \leq \dots \leq N_{f \Pi_m}(r, a_q).$$

By using (4.3), we have

$$m T(r, f) = N_{f \Pi_m}(r, a_j) + O(1), \quad j \geq n,$$

which means

$$(q+1-n)mT(r, f) = \sum_{j=n}^q N_{f^{\text{Um}}}(r, a_j) + O(1) \leq \sum_{j=0}^q N_{f^{\text{Um}}}(r, a_j) + O(1).$$

According to Vojta's idea (cf. [30]), Theorem 4.1 can be translated into a problem in Diophantine approximations. We assume that κ is a number field (i.e., a subfield of $\mathbb{C}$) with an absolute value ρ . The completion of κ relative to the topology induced by ρ is a field which is denoted by κ_ρ . If ρ is Archimedean, then ρ is equivalent to the infinite place $|\cdot|_\infty$, and hence $\kappa_\rho = \mathbb{R}$ or $\mathbb{C}$. If ρ is non-Archimedean, then ρ is equivalent to a p -adic absolute value $|\cdot|_p$, and so κ_ρ is a finite field extension of $\mathbb{Q}_p$. We will denote the normalization of ρ by $\|\cdot\|_\rho$, which is defined for $z \in \kappa$ by

$$(4.4) \quad \|z\|_\rho = \begin{cases} |z|_\infty & : \text{ if } \kappa_\rho = \mathbb{R} \\ |z|_\infty^2 & : \text{ if } \kappa_\rho = \mathbb{C} \\ |z|_p^{[\kappa_\rho:\mathbb{Q}_p]} & : \text{ if } \rho \sim |\cdot|_p \text{ is non-Archimedean.} \end{cases}$$

Under these conventions there exists a set M_κ of inequivalent places of κ satisfying the *product formula*,

$$(4.5) \quad \prod_{\rho \in M_\kappa} \|z\|_\rho = 1, \quad z \in \kappa_*.$$

Let V be a $(n+1)$ -dimensional vector space over κ . Take a base $e = (e_0, \dots, e_n)$ for V . For $\xi = \xi_0 e_0 + \dots + \xi_n e_n \in V$, the norm $\|\xi\|_\rho = \|\xi\|_{\rho, e}$ over the base e is well-defined, and it induces a norm on V^* . Thus the distance from $x = \mathbb{P}(\xi)$ to $\check{E}[a]$ with $a = \mathbb{P}(\alpha) \in \mathbb{P}(V^*)$ is well-defined by

$$0 \leq \|x, a\|_\rho = \frac{\|\langle \xi, \alpha \rangle\|_\rho}{\|\xi\|_\rho \cdot \|\alpha\|_\rho} \leq 1.$$

Take $x \in \mathbb{P}(V)$. Then there exists $\xi \in V_*$ such that $x = \mathbb{P}(\xi)$. The *relative (multiplicative) height* of x is defined by

$$H(x) = \prod_{\rho \in M_\kappa} \|\xi\|_\rho,$$

and the *absolute (logarithmic) height* of x is defined as

$$h(x) = \frac{1}{[\kappa:\mathbb{Q}]} \sum_{\rho \in M_\kappa} \log \|\xi\|_\rho.$$

By the product formula, this does not depend on the choice of ξ . Let S be a finite subset of M_κ containing the set S_∞ of all Archimedean places of κ . Denote by $\mathcal{O}_{\kappa, S}$ the ring of S -integers of κ , i.e.,

$$\mathcal{O}_{\kappa, S} = \{z \in \kappa \mid \|z\|_\rho \leq 1, \rho \notin S\}.$$

A point $\xi = \xi_0 e_0 + \dots + \xi_n e_n \in V$ is said to be a S -integral point if $\xi_i \in \mathcal{O}_{\kappa, S}$ for all $0 \leq i \leq n$. Denote by $\mathcal{O}_{V, S}$ the set of S -integral points of V , that is,

$$\mathcal{O}_{V, S} = \{\xi \in V \mid \|\xi\|_\rho \leq 1, \rho \notin S\}.$$

CONJECTURE 4.3. Take $\varepsilon > 0$. Assume that for $\rho \in S$, the family $\mathcal{A}_\rho = \{a_{\rho,0}, \dots, a_{\rho,q}\} \subset \mathbb{P}(\Pi_m V^*)$ ($q \geq n$) is admissible. Then there exists a finite set $b_1, \dots, b_s$ of $\mathbb{P}(\Pi_m V^*)$ such that the inequality

$$\frac{1}{[\kappa : \mathbb{Q}]} \sum_{\rho \in S} \sum_{j=0}^q \log \frac{1}{\|x^{\Pi_m}, a_{\rho,j}\|_\rho} < m(n+1+\varepsilon)h(x) + O(1)$$

holds for all $x \in \mathbb{P}(V) \setminus \bigcup_i \ddot{E}[b_i]$.

Conjecture 4.3 can be deduced from the following conjecture:

CONJECTURE 4.4. For $\rho \in S$, $i = 0, \dots, n$ take $\alpha_{\rho,i} \in \Pi_m V^* \setminus \{0\}$ such that the system

$$\langle \xi^{\Pi_m}, \alpha_{\rho,i} \rangle = 0, \quad i = 0, \dots, n$$

has only the trivial solution $\xi = 0$ in V . Then for any $\varepsilon > 0$ there exists a finite set $b_1, \dots, b_s$ of $\mathbb{P}(\Pi_m V^*)$ such that the inequality

$$\prod_{\rho \in S} \prod_{i=0}^n \frac{1}{\|\langle \xi^{\Pi_m}, \alpha_{\rho,i} \rangle\|_\rho} \leq \left\{ \max_{\rho \in S} \|\xi\|_\rho \right\}^\varepsilon$$

holds for all S -integral points $\xi \in \mathcal{O}_{V,S} \setminus \bigcup_i E[b_i]$.

We also suggest the following problem:

CONJECTURE 4.5. Take $\varepsilon > 0$. Assume that the family $\mathcal{A} = \{a_0, \dots, a_q\} \subset \mathbb{P}(\Pi_m V^*)$ ($q \geq n$) is admissible. Then the set of points of $\mathbb{P}(V) \setminus \bigcup_j \ddot{E}[a_j]$ satisfying

$$\sum_{j=0}^q m_{x^{\Pi_m}}(a_j) \geq m(2n+\varepsilon)h(x) + O(1)$$

is finite.

For the case $m = 1$, the conjectures 4.3, 4.4 and 4.5 are well-known (cf. [25]).

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