Simplified models of superconducting-normal-superconducting junctions and their numerical approximations[†]

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When a thin layer of normal (non-superconducting) material is placed between layers of superconducting material, a superconducting-normal-superconducting junction is formed. This paper considers a model for the junction based on the Ginzburg–Landau equations as the thickness of the normal layer tends to zero. The model is first derived formally by averaging the unknown variables in the normal layer. Rigorous convergence is then established, as well as an estimate for the order of convergence. Numerical results are shown for one-dimensional junctions.

1 Introduction

Samples consisting of both non-superconducting (normal) and superconducting materials are of interest in both the study of superconducting phenomena and the design of superconducting devices including SQUIDS [1]. A sample in which a layer of normal material is sandwiched between two layers of superconducting material is called an SNS junction or Josephson junction, and is well known for its quantum mechanical Josephson effects. In this paper, we take advantage of the fact that the normal layer is very thin to derive a simplified model within the framework of the phenomenological Ginzburg–Landau theory.

The asymptotic behaviour of the solutions of the Ginzburg–Landau equations as the layer thickness tending to zero has been much studied in the literature. In Chapman & Gunzburger [2], a thin superconducting film is studied. For superconducting-normal-superconducting junctions, discussions of supercurrent across the junction have been made by Chapman *et al.* [3] for one-dimensional junctions. In Hoffmann *et al.* [4], more detailed studies were made for junctions that have very thin normal layer. Nevertheless, in the leading order simplifications, the model equations derived in Hoffmann *et al.* [4] remain a system of coupled nonlinear equations for variables in both the superconducting and normal layers. It is the purpose of this paper to show that, to leading order, the thin

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normal layer induces negligible influence on the device and thus the resulting equations could be even simpler. Higher order corrections, rigorous convergence theory for the steady state models, analysis on the order of convergence for the time-dependent models as well as numerical simulations are also presented here. Further investigations of the Josephson *d.c.* and *a.c.* effects may also be carried out with the help of the simplified models and the rigorous analysis developed in this paper.

The rest of the paper is organized as follows. In §2, we describe the Ginzburg–Landau type models for superconducting-normal-superconducting junctions. In §3, a formal derivation of the simplified model is given in the steady state case and a rigorous convergence proof is given in §4. The time-dependent equations are considered in §5, with the detailed estimate on the order of convergence to the leading order equations given in the appendix. A brief discussion of the high- κ , high field limit is given in the §6. Numerical approximations and computational experiments are given in §7, and final remarks are given in §8.

2 Ginzburg–Landau free energy

We begin this section by introducing some notation that will be used below. Throughout, for any non-negative integer k and domain $\mathcal{D} \subset \mathbb{R}^n$, $n = 2$ or 3 , $H^k(\mathcal{D})$ will denote the Sobolev space of real-valued functions having square-integrable derivatives of order up to k . The corresponding spaces of complex-valued functions will be denoted by $\mathcal{H}^k(\mathcal{D})$. Corresponding spaces of vector-valued functions, each of whose components belong to $H^k(\mathcal{D})$, will be denoted by $\mathbf{H}^k(\mathcal{D})$, i.e. $\mathbf{H}^k(\mathcal{D}) = [H^k(\mathcal{D})]^n$. Norms of functions belonging to $H^k(\mathcal{D})$, $\mathbf{H}^k(\mathcal{D})$, and $\mathcal{H}^k(\mathcal{D})$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_{k,\mathcal{D}}$ or $\|\cdot\|_k$. The latter notation will be used when there is no chance of confusion. For details concerning these spaces, one may consult Adams [5]. We will also use

$$\mathbf{H}(\mathbb{R}^3) = \text{the closure of } \{\mathbf{A} \in \mathbf{C}_0^\infty(\mathbb{R}^3) \mid \|\mathbf{A}\|^2 = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx dy dz\},$$

and

$$\mathbf{H}(\text{div}, \mathbb{R}^3) = \{\mathbf{A} \in \mathbf{H}(\mathbb{R}^3) : \text{div} \mathbf{A} = 0\}.$$

To briefly describe the background, we start with the Ginzburg–Landau theory of superconductivity [6, 7]. The non-dimensional Gibbs free energy postulated by Ginzburg and Landau for a superconducting sample Ω , up to an unimportant constant, is given by

$$\begin{aligned} \mathcal{G}(\Psi, \mathbf{A}) = & \int_{\Omega} \left(\frac{1}{2}(|\Psi|^2 - 1)^2 + \left| \frac{i}{\kappa} \nabla \Psi + \mathbf{A} \Psi \right|^2 \right) dx dy dz \\ & + \int_{\mathbb{R}^3} |\mathbf{h} - \mathbf{H}_{\text{ext}}|^2 dx dy dz, \end{aligned}$$

where Ψ is the complex order parameter such that $|\Psi|^2$ represents the number density of superconducting electrons ($|\Psi| = 1$ corresponds to the superconducting state, $|\Psi| = 0$ corresponds to the normal state). $\mathbf{A}$ is the vector magnetic potential and $\mathbf{h} = \text{curl} \mathbf{A}$. $\mathbf{H}_{\text{ext}}$ is the constant external field and κ is a material parameter which determines the type of superconducting material. Detailed studies of the Ginzburg–Landau theory may be found in Tinkham [1].

FIGURE 1. An illustration of an SNS junction with normal layer having constant thickness.

In this paper, we consider a three-dimensional SNS junction occupying the domain Ω which is symmetric with respect to the (x, y) -plane given by

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : (x, y) \in D, -d(x, y) < z < d(x, y) \right\},$$

where D is a bounded region in $\mathbb{R}^2$ with smooth boundary and $d(x, y) \geq d_0(x, y) > 0$ is the thickness of the junction which is assumed to be a smooth function for all $(x, y) \in D$. The superconducting-normal-superconducting (SNS) junction refers to a junction Ω formed by two superconducting layers Ω_{S_m} , ($m = 1, 2$), and a normal layer Ω_N sandwiched in between. More specifically,

$$\begin{aligned} \Omega_N &= \left\{ (x, y, z) \in \mathbb{R}^3 : (x, y) \in D, -\epsilon d_0(x, y) < z < \epsilon d_0(x, y) \right\}, \\ \Omega_{S_1} &= \left\{ (x, y, z) \in \mathbb{R}^3 : (x, y) \in D, \epsilon d_0(x, y) < z < d(x, y) \right\}, \\ \Omega_{S_2} &= \left\{ (x, y, z) \in \mathbb{R}^3 : (x, y) \in D, -d(x, y) < z < -\epsilon d_0(x, y) \right\} \end{aligned}$$

for some small positive parameter ϵ .

The regions are illustrated in Figure 1, where the function d_0 is taken to be a constant. For an SNS junction, the (non-dimensional) Gibbs free energy, up to an unimportant additive constant, is modified to [3]

$$\begin{aligned} \mathcal{G}_\epsilon(\Psi_\epsilon, \mathbf{A}_\epsilon) &= \sum_{m=1}^2 \int_{\Omega_{S_m}} \left(\frac{1}{2}(|\Psi_\epsilon|^2 - 1)^2 + \left| \frac{i}{\kappa} \nabla \Psi_\epsilon + \mathbf{A}_\epsilon \Psi_\epsilon \right|^2 \right) dx dy dz \\ &\quad + \int_{\Omega_N} \left(\frac{\alpha}{2} |\Psi_\epsilon|^2 + \left| \frac{i}{\kappa} \nabla \Psi_\epsilon + \mathbf{A}_\epsilon \Psi_\epsilon \right|^2 \right) dx dy dz \\ &\quad + \int_{\mathbb{R}^3} |\mathbf{h} - \mathbf{H}_{ext}|^2 dx dy dz. \end{aligned}$$

Here, Ψ_ϵ is the complex superconducting order parameter, $\mathbf{A}_\epsilon$ is the vector magnetic potential, and $\alpha > 0$ is a constant determined by the material in the normal layer, Ω_N . For detailed discussion on the value of α for various junctions, we refer to Chapman *et al.* [3]. The main concern of this paper is the limit as the thickness of the normal layer Ω_N approaches zero (i.e. $\epsilon \rightarrow 0$). Related discussions have been given in Chapman *et al.* [3] and Hoffmann *et al.* [4].

The minimizer of $\mathcal{G}_\epsilon$ satisfies the following dimensionless Ginzburg–Landau equations which hold in Ω [3]:

$$\left(\frac{1}{\kappa} \nabla - i \mathbf{A}_\epsilon \right)^2 \Psi_\epsilon = (|\Psi_\epsilon|^2 - 1) \Psi_\epsilon \quad \text{in } \Omega_{S_m}, \quad m = 1, 2, \quad (2.1)$$

$$\left(\frac{1}{\kappa}\nabla - i\mathbf{A}_\epsilon\right)^2 \Psi_\epsilon = \alpha \Psi_\epsilon \quad \text{in } \Omega_N, \quad (2.2)$$

$$\begin{aligned} (\text{curl})^2 \mathbf{A}_\epsilon &= \frac{i}{2\kappa} (\Psi_\epsilon \nabla \Psi_\epsilon^* - \Psi_\epsilon^* \nabla \Psi_\epsilon) \\ &\quad - |\Psi_\epsilon|^2 \mathbf{A}_\epsilon, \quad \text{in } \Omega_{S_1} \cup \Omega_N \cup \Omega_{S_2}, \end{aligned} \quad (2.3)$$

$$(\text{curl})^2 \mathbf{A}_\epsilon = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \Omega, \quad (2.4)$$

with boundary conditions

$$[\text{curl } \mathbf{A}_\epsilon]_{\partial\Omega} = \mathbf{0}, \quad (2.5)$$

$$[\mathbf{A}_\epsilon]_{\partial\Omega} = \mathbf{0}, \quad (2.6)$$

$$\mathbf{n} \cdot \left(\frac{1}{\kappa}\nabla - i\mathbf{A}_\epsilon\right) \Psi_\epsilon = 0 \quad \text{on } \partial\Omega, \quad (2.7)$$

$$\text{curl } \mathbf{A}_\epsilon \rightarrow \mathbf{H}_{ext} \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad (2.8)$$

and interface conditions

$$[\mathbf{A}_\epsilon]_{\Gamma_m} = \mathbf{0}, [\Psi_\epsilon]_{\Gamma_m} = 0, \quad \text{for } m = 1, 2, \quad (2.9)$$

$$\left[\left(\frac{1}{\kappa}\nabla - i\mathbf{A}_\epsilon\right) \Psi_\epsilon \cdot \mathbf{n}\right]_{\Gamma_m} = 0, \quad \text{for } m = 1, 2, \quad (2.10)$$

$$[\text{curl } \mathbf{A}_\epsilon \times \mathbf{n}]_{\Gamma_m} = \mathbf{0}, \quad \text{for } m = 1, 2. \quad (2.11)$$

Here, the interface is given by $\Gamma_m = \overline{\Omega}_{S_m} \cap \overline{\Omega}_N$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, $[\cdot]_\Gamma$ denotes the jump in the enclosed quantity across the boundary, Γ , and $*$ denotes complex conjugation.

Using the gauge transformation from Du *et al.* [7], $\mathbf{Q} = \mathbf{A}_\epsilon - \nabla\chi/\kappa$, $\Psi_\epsilon = f e^{i\chi}$ with f , χ real ($\chi = 0$ in $\mathbb{R}^3 \setminus \Omega$) equations (1)–(11) can be rewritten in a simpler form:

$$\frac{1}{\kappa^2} \nabla^2 f = f^3 - f + f |\mathbf{Q}|^2 \quad \text{in } \Omega_{S_m}, \quad m = 1, 2, \quad (2.12)$$

$$\frac{1}{\kappa^2} \nabla^2 f = \alpha f + f |\mathbf{Q}|^2 \quad \text{in } \Omega_N, \quad (2.13)$$

$$\text{div}(f^2 \mathbf{Q}) = 0 \quad \text{in } \Omega_{S_1} \cup \Omega_N \cup \Omega_{S_2}, \quad (2.14)$$

$$(\text{curl})^2 \mathbf{Q} = -f^2 \mathbf{Q} \quad \text{in } \Omega_{S_1} \cup \Omega_N \cup \Omega_{S_2}, \quad (2.15)$$

$$(\text{curl})^2 \mathbf{Q} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \Omega, \quad (2.16)$$

$$\mathbf{H} = \text{curl } \mathbf{Q}, \quad \text{in } \Omega_{S_1} \cup \Omega_N \cup \Omega_{S_2}, \quad (2.17)$$

with boundary conditions

$$[\text{curl } \mathbf{Q}]_{\partial\Omega} = \mathbf{0}, \quad (2.18)$$

$$[\mathbf{Q}]_{\partial\Omega} = \mathbf{0}, \quad (2.19)$$

$$\frac{\partial f}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \quad (2.20)$$

$$\mathbf{H} \rightarrow \mathbf{H}_{ext} \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad (2.21)$$

and interface conditions

$$[\mathbf{Q}]_{\Gamma_m} = \mathbf{0}, \quad [f]_{\Gamma_m} = 0, \quad \text{for } m = 1, 2, \quad (2.22)$$

$$[\text{curl } \mathbf{Q} \times \mathbf{n}]_{\Gamma_m} = \mathbf{0}, \quad \text{for } m = 1, 2, \quad (2.23)$$

$$\left[\frac{\partial f}{\partial \mathbf{n}} \right]_{\Gamma_m} = \mathbf{0}, \quad \text{for } m = 1, 2. \quad (2.24)$$

3 Formal derivation of the model

We consider the formal asymptotic limit of the solutions to the SNS junction equations as $\epsilon \rightarrow 0$ with κ fixed. The existence of various limits and the validity of the limiting processes which arise in this derivation will be justified in the next section.

3.1 Leading order equation

To determine the effect of the middle normal layer, we use the approach in Chapman *et al.* [2], where the authors develop a model for a thin superconducting film by considering the average value of the appropriate equations.

To determine the leading order equations, we will use the scaling

$$\tilde{z} = d(z - \epsilon d_0)/(d - \epsilon d_0) \quad \text{on} \quad \Omega_{S_1},$$

and define

$$\Omega_{S_1}^{(0)} = \{(x, y, \tilde{z}) : 0 < \tilde{z} < d(x, y)\}.$$

By the change of variables $z \rightarrow \tilde{z}$ in (2.12), (2.14) and (2.15) and letting $\epsilon \rightarrow 0$, we get the following equations:

$$\frac{1}{\kappa^2} \nabla^2 f^{(0)} = (f^{(0)})^3 - f^{(0)} + f^{(0)} |\mathbf{Q}^{(0)}|^2 \quad \text{in } \Omega_{S_1}^{(0)}, \quad (3.1)$$

$$\text{div}((f^{(0)})^2 \mathbf{Q}^{(0)}) = 0 \quad \text{in } \Omega_{S_1}^{(0)}, \quad (3.2)$$

$$(\text{curl})^2 \mathbf{Q}^{(0)} = -(f^{(0)})^2 \mathbf{Q}^{(0)} \quad \text{in } \Omega_{S_1}^{(0)}, \quad (3.3)$$

where $(f^{(0)}, \mathbf{Q}^{(0)})$ denote the leading order behaviour of $(f, \mathbf{Q})$. Note that by a similar scaling argument we find that $(f^{(0)}, \mathbf{Q}^{(0)})$ satisfy (3.1)–(3.3) on the scaled domain:

$$\Omega_{S_2}^{(0)} = \{(x, y, \tilde{z}) : -d(x, y) < \tilde{z} < 0\}.$$

In this section, we define $(f_{S_m}, \mathbf{Q}_{S_m}) = (f, \mathbf{Q})$ for $(x, y, z) \in \Omega_{S_m}$ and $m = 1, 2$, and $(f_N, \mathbf{Q}_N) = (f, \mathbf{Q})$ for $(x, y, z) \in \Omega_N$. The average value of a function g is defined by

$$\bar{g} = \frac{1}{2\epsilon d_0} \int_{-\epsilon d_0}^{\epsilon d_0} g dz.$$

Now, if our leading order solutions are smooth across the interface of the two new regions $\Omega_{S_1}^{(0)}$ and $\Omega_{S_2}^{(0)}$, then the conventional Ginzburg–Landau equations will be satisfied and normal layer has negligible effect to the leading order.

Integrating (2.13) from $-\epsilon d_0$ to ϵd_0 , we obtain:

$$\frac{1}{2\kappa^2 \epsilon d_0} \left[\frac{\partial f_N}{\partial z} \right]_{-\epsilon d_0}^{\epsilon d_0} + \overline{\nabla^2 f_N} = \overline{\alpha f_N} + \overline{f_N |\mathbf{Q}_N|^2},$$

where

$$\bar{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).$$

Assuming f_N , $\mathbf{Q}_N$, and $\bar{\nabla}^2 f_N$ are bounded, we have

$$\left. \frac{\partial f_N}{\partial z} \right|_{-\epsilon d_0}^{\epsilon d_0} = O(\epsilon).$$

Using the interface condition (2.24) we have

$$\frac{\partial f_{S_m}}{\partial n} = -\frac{\partial f_N}{\partial z} \pm \epsilon \bar{\nabla} f_N \cdot \nabla d_0, \quad \text{on } \Gamma_m, \quad m = 1, 2$$

which implies

$$\begin{aligned} \left. \frac{\partial f_{S_1}}{\partial n} \right|_{\epsilon d_0} - \left. \frac{\partial f_{S_2}}{\partial n} \right|_{-\epsilon d_0} &= - \left[\left. \frac{\partial f_N}{\partial z} \right|_{-\epsilon d_0}^{\epsilon d_0} \right] + \epsilon \nabla d_0 \cdot \left(\bar{\nabla} f_N \Big|_{\epsilon d_0} + \bar{\nabla} f_N \Big|_{-\epsilon d_0} \right) \\ &= O(\epsilon) \end{aligned}$$

So, as $\epsilon \rightarrow 0$,

$$\left. \frac{\partial f_{S_1}^{(0)}}{\partial z} \right|_{0+} = \left. \frac{\partial f_{S_2}^{(0)}}{\partial z} \right|_{0-}. \quad (3.4)$$

We now consider the interface condition for $\mathbf{Q}$. We use (2.17) and write out (2.15) in component form as:

$$\left(\frac{\partial h_{N,3}}{\partial y} - \frac{\partial h_{N,2}}{\partial z} \right) = -f_N^2 q_{N,1}, \quad (3.5)$$

$$\left(\frac{\partial h_{N,1}}{\partial z} - \frac{\partial h_{N,3}}{\partial x} \right) = -f_N^2 q_{N,2}, \quad (3.6)$$

$$\left(\frac{\partial h_{N,2}}{\partial x} - \frac{\partial h_{N,1}}{\partial y} \right) = -f_N^2 q_{N,3}, \quad (3.7)$$

where $h_{N,j}$ and $q_{N,j}$ represent the j th component of $\mathbf{H}_N$ and $\mathbf{Q}_N$, respectively. Also, since $\mathbf{H}_N$ is divergence free:

$$\frac{\partial h_{N,1}}{\partial x} + \frac{\partial h_{N,2}}{\partial y} + \frac{\partial h_{N,3}}{\partial z} = 0. \quad (3.8)$$

Integrating equations (3.5)–(3.6) and (3.8) from $-\epsilon d_0$ to ϵd_0 , we have

$$\begin{aligned} \frac{-1}{2\epsilon d_0} [h_{N,2}]_{-\epsilon d_0}^{\epsilon d_0} &= \overline{f_N^2 q_{N,1}} - \frac{\overline{\partial h_{N,3}}}{\partial y}, \\ \frac{1}{2\epsilon d_0} [h_{N,1}]_{-\epsilon d_0}^{\epsilon d_0} &= \overline{f_N^2 q_{N,2}} + \frac{\overline{\partial h_{N,3}}}{\partial x}, \\ \frac{1}{2\epsilon d_0} [h_{N,3}]_{-\epsilon d_0}^{\epsilon d_0} &= -\frac{\overline{\partial h_{N,1}}}{\partial x} - \frac{\overline{\partial h_{N,2}}}{\partial y}. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we have

$$\mathbf{H}_N^{(0)} \Big|_{-\epsilon d_0}^{\epsilon d_0} = O(\epsilon).$$

Using the interface conditions (2.23), this implies

$$\operatorname{curl} \mathbf{Q}_{S_1}^{(0)} \times \mathbf{n} \Big|_{\epsilon d_0} - \operatorname{curl} \mathbf{Q}_{S_2}^{(0)} \times \mathbf{n} \Big|_{-\epsilon d_0} = O(\epsilon),$$

or, as $\epsilon \rightarrow 0$,

$$\operatorname{curl} \mathbf{Q}_{S_1}^{(0)} \times \mathbf{k} \Big|_{0+} = \operatorname{curl} \mathbf{Q}_{S_2}^{(0)} \times \mathbf{k} \Big|_{0-}, \quad (3.9)$$

where $\mathbf{k}$ denotes the unit vector in the z -direction. So the leading order equations are (dropping the superscript (0)):

$$\frac{1}{\kappa^2} \nabla^2 f = f^3 - f + f |\mathbf{Q}|^2 \quad \text{in } \Omega_{S_m}^{(0)}, \quad m = 1, 2, \quad (3.10)$$

$$\operatorname{div} (f^2 \mathbf{Q}) = 0 \quad \text{in } \Omega_{S_m}^{(0)}, \quad m = 1, 2, \quad (3.11)$$

$$(\operatorname{curl})^2 \mathbf{Q} = -f^2 \mathbf{Q} \quad \text{in } \Omega_{S_m}^{(0)}, \quad m = 1, 2, \quad (3.12)$$

$$(\operatorname{curl})^2 \mathbf{Q} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \Omega, \quad (3.13)$$

$$\mathbf{H} = \operatorname{curl} \mathbf{Q}, \quad \text{in } \Omega_{S_m}^{(0)}, \quad m = 1, 2, \quad (3.14)$$

with boundary conditions

$$[\operatorname{curl} \mathbf{Q}]_{\partial \Omega} = \mathbf{0}, \quad [\mathbf{Q}]_{\partial \Omega} = 0, \quad (3.15)$$

$$\frac{\partial f}{\partial \mathbf{n}} = 0 \quad \text{on } \partial \Omega, \quad m = 1, 2, \quad (3.16)$$

$$\mathbf{H} \rightarrow \mathbf{H}_{ext} \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad (3.17)$$

and interface conditions

$$[\mathbf{Q}]_{z=0} = 0, \quad [f]_{z=0} = 0, \quad (3.18)$$

$$\left[\frac{\partial f}{\partial z} \right]_{z=0} = 0, \quad (3.19)$$

$$[\operatorname{curl} \mathbf{Q} \times \mathbf{k}]_{z=0} = \mathbf{0}. \quad (3.20)$$

Remark 1. The above equations imply that, to leading order, there is no coupling between the solutions of the superconducting layers and the normal layer as the normal layer thickness goes to zero (since $\epsilon \rightarrow 0$). So, the SNS junction is not affected by the properties of middle normal layer in this limit, at least to the leading order.

3.2 Next order correction

Having determined $f^{(0)}$ and $\mathbf{Q}^{(0)}$ from the leading order equations, we may determine the next order correction in equations (13)–(18) by expanding

$$\begin{aligned} f &= f^{(0)} + \epsilon f^{(1)}, \\ \mathbf{Q} &= \mathbf{Q}^{(0)} + \epsilon \mathbf{Q}^{(1)}. \end{aligned}$$

It can be shown that $f^{(1)}$ and $\mathbf{Q}^{(1)}$ satisfy the following equations in $\Omega_{S_m}^{(0)}$ for $m = 1, 2$.

$$\begin{aligned} \frac{1}{\kappa^2} \nabla^2 f^{(1)} &= f^{(1)} (3|f^{(0)}|^2 - 1 + |\mathbf{Q}^{(0)}|^2) + 2\mathbf{Q}^{(1)} \cdot f^{(0)} \mathbf{Q}^{(0)}, \\ \nabla f^{(0)} \cdot (2f^{(1)} \mathbf{Q}^{(0)} + f^{(0)} \mathbf{Q}^{(1)}) &= 0, \end{aligned}$$

$$\begin{aligned}\operatorname{curl}^2 \mathbf{Q}^{(1)} &= -f^{(0)} (2f^{(1)} \mathbf{Q}^{(0)} + f^{(0)} \mathbf{Q}^{(1)}), \\ \mathbf{H}^{(1)} &= \operatorname{curl} \mathbf{Q}^{(1)}, \\ \operatorname{curl}^2 \mathbf{Q}^{(1)} &= \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \Omega,\end{aligned}$$

with boundary conditions

$$\begin{aligned}[\operatorname{curl} \mathbf{Q}^{(1)}]_{\partial\Omega} &= \mathbf{0}, \\ [\mathbf{Q}^{(1)}]_{\partial\Omega} &= 0, \\ \frac{\partial f^{(1)}}{\partial n} &= 0 \quad \text{on } \partial\Omega, \\ \mathbf{H}^{(1)} &\rightarrow \mathbf{0} \quad \text{as } |\mathbf{x}| \rightarrow \infty,\end{aligned}$$

and interface conditions

$$\begin{aligned}[f^{(1)}]_{z=0} &= 0, \quad [\mathbf{Q}^{(1)}]_{z=0} = 0, \\ \left[\frac{\partial f^{(1)}}{\partial z} \right]_{z=0} &= 2\kappa^2 d_0 (\alpha + 1) f^{(0)}(x, y, 0) - 2\kappa^2 d_0 (f^{(0)})^3(x, y, 0), \\ [\operatorname{curl} \mathbf{Q}^{(1)} \times \mathbf{k}]_{z=0} &= 2d_0 \left(-\frac{\partial h_3^{(0)}}{\partial y} + (f^{(0)})^2 q_1^{(0)}, \frac{\partial h_3^{(0)}}{\partial x} + (f^{(0)})^2 q_2^{(0)}, 0 \right)^T.\end{aligned}$$

Remark 2. The above equations illustrate how the SNS junction might be affected by the properties of middle normal layer in the next order correction when the normal layer thickness goes to zero. The material parameter for the normal layer, α , appears in the interface conditions.

4 Rigorous proof of convergence

We use ideas similar to those presented by Chapman *et al.* [2] to show that the derivations made in the previous section are rigorous.

4.1 An equivalent formulation

Similar to Chapman *et al.* [2], define $\mathbf{B} = \mathbf{B}(x, y, z)$ such that

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega, \quad (4.1)$$

$$\operatorname{curl} \mathbf{B} = \mathbf{H}_{\text{ext}} \quad \text{in } \mathbb{R}^3, \quad (4.2)$$

(e.g. for $\mathbf{H}_{\text{ext}} = (H_1, H_2, H_3)$ with $H_i = \text{constant}$ for $i=1,2,3$, then $\mathbf{B} = (zH_2, xH_3, yH_1)$ satisfies (4.1)–(4.2)). We can then define an equivalent formulation of the free energy:

$$\begin{aligned}\tilde{\mathcal{G}}_\epsilon(\Psi, \mathbf{A}) &= \sum_{m=1}^2 \int_{\Omega_{S_m}} \frac{1}{2} (|\Psi|^2 - 1)^2 + \left| \frac{i}{\kappa} \nabla \Psi + \mathbf{A} \Psi + \mathbf{B} \Psi \right|^2 dx dy dz \\ &\quad + \int_{\Omega_N} \frac{\alpha}{2} |\Psi|^2 + \left| \frac{i}{\kappa} \nabla \Psi + \mathbf{A} \Psi + \mathbf{B} \Psi \right|^2 dx dy dz \\ &\quad + \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A}|^2 dx dy dz.\end{aligned} \quad (4.3)$$

For convenience, we also use

$$\begin{aligned}\tilde{\mathcal{G}}(\Psi, \mathbf{A}) &= \sum_{m=1}^2 \int_{\Omega_{S_m}} \left(\frac{1}{2}(|\Psi|^2 - 1)^2 + \left| \frac{i}{\kappa} \nabla \Psi + \mathbf{A} \Psi + \mathbf{B} \Psi \right|^2 \right) dx dy dz \\ &\quad + \int_{\Omega_N} \left(\frac{1}{2}(|\Psi|^2 - 1)^2 + \left| \frac{i}{\kappa} \nabla \Psi + \mathbf{A} \Psi + \mathbf{B} \Psi \right|^2 \right) dx dy dz \\ &\quad + \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A}|^2 dx dy dz.\end{aligned}$$

We consider the following variational problem

$$(I) \quad \min_{\Psi \in \mathbf{H}^1(\Omega), \mathbf{A} \in \mathbf{H}(\mathbb{R}^3)} \tilde{\mathcal{G}}_\epsilon(\Psi, \mathbf{A}).$$

Using a gauge transformation, the above problem is equivalent to

$$(II) \quad \min_{\Psi \in \mathbf{H}^1(\Omega), \mathbf{A} \in \mathbf{H}(\operatorname{div}, \mathbb{R}^3)} \tilde{\mathcal{G}}_\epsilon(\Psi, \mathbf{A}).$$

Let $(\Psi_\epsilon, \mathbf{A}_\epsilon)$ denote a minimizer of $\tilde{\mathcal{G}}_\epsilon(\Psi, \mathbf{A})$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}(\operatorname{div}, \mathbb{R}^3)$. Note that $(\Psi_\epsilon, \mathbf{A}_\epsilon + \mathbf{B})$ is then a minimizer of $\mathcal{G}_\epsilon(\Psi, \mathbf{A})$.

4.2 Uniform bounds

Here, we derive some uniform bounds, independent of ϵ , for the minimizer $(\Psi_\epsilon, \mathbf{A}_\epsilon)$. First, since $(\Psi, \mathbf{A}) = (0, \mathbf{0})$ is feasible for (II), we get:

Lemma 1 *For any $\epsilon > 0$,*

$$\tilde{\mathcal{G}}_\epsilon(\Psi_\epsilon, \mathbf{A}_\epsilon) \leq \int_D d(x, y) dx dy.$$

Thus, using the norm equivalence $\|\operatorname{curl} \mathbf{A}\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 = \|\nabla \mathbf{A}\|_{\mathbf{L}^2(\mathbb{R}^3)}^2$ in $\mathbf{H}(\operatorname{div}, \mathbb{R}^3)$, we get:

Corollary 1 *For any $\epsilon > 0$,*

$$\|\nabla \mathbf{A}_\epsilon\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq \left(\int_D d(x, y) dx dy \right)^{1/2}.$$

Using a standard interpolation inequality [8]

$$\|\mathbf{A}\|_{\mathbf{L}^6(\mathbb{R}^3)} \leq c \|\nabla \mathbf{A}\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

we get,

Corollary 2 *There exists a constant $c > 0$, such that for any $\epsilon > 0$,*

$$\|\mathbf{A}_\epsilon\|_{\mathbf{L}^6(\mathbb{R}^3)} \leq c.$$

By the imbedding $\mathbf{L}^6(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ we get from Corollaries 1 and 2 that,

Proposition 1 *There exists a constant $c > 0$, independent of ϵ , such that:*

$$\|\mathbf{A}_\epsilon\|_{\mathbf{H}^1(\Omega)} \leq c.$$

Next we consider the order parameter. Similar to [4], we can show that:

Lemma 2

$$\|\Psi_\epsilon\|_{\mathcal{L}^\infty(\Omega)} \leq 1.$$

By Lemmas 1 and 2, and Proposition 1 we get:

Lemma 3 *There exists a constant $c > 0$, independent of ϵ such that:*

$$\|\nabla \Psi_\epsilon\|_{\mathcal{L}^2(\Omega)} \leq c.$$

In turn, we get

Proposition 2 *There exists a constant $c > 0$, independent of ϵ , such that:*

$$\|\Psi_\epsilon\|_{\mathcal{H}^1(\Omega)} \leq c.$$

Using the above estimates, the standard elliptic regularity results and interpolation inequalities, we can also get uniform estimates on the higher order derivatives of $(\Psi_\epsilon, \mathbf{A}_\epsilon)$.

4.3 Passing to the limit

From the above estimates and compactness results in Sobolev spaces, we have

Theorem 1 *There exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$ such that*

$$\begin{aligned} \Psi_{\epsilon_k} &\rightharpoonup \overline{\Psi} \quad \text{in } \mathcal{H}^1(\Omega), \\ \mathbf{A}_{\epsilon_k} &\rightharpoonup \overline{\mathbf{A}} \quad \text{in } \mathbf{H}^1(\Omega) \cap \mathbf{H}(\mathbb{R}^3), \\ \Psi_{\epsilon_k} &\rightarrow \overline{\Psi} \quad \text{in } \mathcal{L}^r(\Omega), \\ \mathbf{A}_{\epsilon_k} &\rightarrow \overline{\mathbf{A}} \quad \text{in } \mathbf{L}^r(\Omega), \end{aligned}$$

where $2 \leq r < 6$.

In fact, by the elliptic regularity, we may strengthen the H^1 weak convergence in Theorem 1 to strong convergence by deriving a few additional estimates. Details are omitted here.

Since $|\Psi_{\epsilon_k}| \leq 1$ and $\text{vol}(\Omega_N) = O(\epsilon_k)$ we have

$$\left| \int_{\Omega_N} \left(\frac{\alpha}{2} |\Psi_{\epsilon_k}|^2 - \frac{1}{2} (|\Psi_{\epsilon_k}|^2 - 1)^2 \right) dx dy dz \right| \leq c\epsilon_k,$$

for some constant c independent of ϵ_k . Therefore, we get

$$\lim_{\epsilon_k \rightarrow 0} |\tilde{\mathcal{G}}_{\epsilon_k}(\Psi_{\epsilon_k}, \mathbf{A}_{\epsilon_k}) - \mathcal{G}(\overline{\Psi}, \overline{\mathbf{A}})| = 0.$$

Since it is not difficult to check that the functional $\mathcal{G}$ is weakly lower semi-continuous in $\mathcal{H}^1(\Omega) \times \mathbf{H}(\mathbb{R}^3)$. We finally get

$$\lim_{\epsilon_k \rightarrow 0} \tilde{\mathcal{G}}_{\epsilon_k}(\Psi_{\epsilon_k}, \mathbf{A}_{\epsilon_k}) \geq \mathcal{G}(\bar{\Psi}, \bar{\mathbf{A}}) \quad .$$

We now show that

Theorem 2 $(\bar{\Psi}, \bar{\mathbf{A}})$ is a minimizer of $\mathcal{G}$.

Proof Suppose that there exists a minimizer $(\zeta, \mathbf{Q})$ of $\mathcal{G}$ such that

$$\mathcal{G}(\bar{\Psi}, \bar{\mathbf{A}}) > \mathcal{G} + \delta/2 = \mathcal{G}(\zeta, \mathbf{Q}) + \delta$$

for some $\delta > 0$. Similar to lemma 2, we have that

$$\|\zeta\|_{\mathcal{L}^\infty(\Omega)} \leq 1.$$

Then for k large, we have

$$\tilde{\mathcal{G}}_{\epsilon_k}(\zeta, \mathbf{Q}) - \mathcal{G}(\zeta, \mathbf{Q}) = \int_{\Omega_N} \left(\frac{\alpha}{2} |\zeta|^2 - \frac{1}{2} (|\zeta|^2 - 1)^2 \right) dx dy dz < \delta/2.$$

since $vol(\Omega_N) \leq c\epsilon_k$. Meanwhile, for ϵ_k small, we may also have

$$\tilde{\mathcal{G}}_{\epsilon_k}(\Psi_{\epsilon_k}, \mathbf{A}_{\epsilon_k}) - \mathcal{G}(\bar{\Psi}, \bar{\mathbf{A}}) > -\delta/2.$$

So,

$$\begin{aligned} \tilde{\mathcal{G}}_{\epsilon_k}(\Psi_{\epsilon_k}, \mathbf{A}_{\epsilon_k}) &> \mathcal{G}(\bar{\Psi}, \bar{\mathbf{A}}) - \delta/2 > \mathcal{G}(\zeta, \mathbf{Q}) + \delta/2 \\ &> \tilde{\mathcal{G}}_{\epsilon_k}(\zeta, \mathbf{Q}) \geq \min \tilde{\mathcal{G}}_{\epsilon_k}. \end{aligned}$$

This is a contradiction. □

The above theorem illustrates that the leading order approximation of $(\Psi_{\epsilon_k}, \mathbf{A}_{\epsilon_k})$ is given by the minimizer of $\mathcal{G}$ which, in the appropriate gauge, satisfies the Euler–Lagrange equations given in (2.1–2.4) with the boundary and interface conditions (2.6–2.11).

5 Time-dependent equations

The time-dependent Ginzburg–Landau equations are often used to study the motion of vortices in type-II superconductors [10]. More importantly, they provide a set of model equations for the rigorous investigation of the *a.c.* and *d.c.* Josephson effects. The equations for the SNS junctions developed in Chapman *et al.* [3] were generalized to the time-dependent case in Hoffmann *et al.* [4]. For simplicity, we ignore the effect of the exterior of Ω . From Hoffmann *et al.* [4], under an appropriate gauge transformation, the time dependent equations are given by:

$$\left(\frac{\partial \Psi_\epsilon}{\partial t} - i\kappa \Psi_\epsilon \nabla \cdot \mathbf{A}_\epsilon \right) - (\nabla - i\kappa \mathbf{A}_\epsilon)^2 \Psi_\epsilon = \Psi_\epsilon (1 - |\Psi_\epsilon|^2), \quad \text{in } \Omega_{S_m T}, \quad m = 1, 2, \quad (5.1)$$

$$\left(\frac{\partial \Psi_\epsilon}{\partial t} - i\kappa \Psi_\epsilon \nabla \cdot \mathbf{A}_\epsilon \right) - (\nabla - i\kappa \mathbf{A}_\epsilon)^2 \Psi_\epsilon = \alpha \Psi_\epsilon \quad \text{in } \Omega_{NT}, \quad (5.2)$$

$$\frac{\partial \mathbf{A}_\epsilon}{\partial t} - \nabla^2 \mathbf{A}_\epsilon + |\Psi_\epsilon|^2 \mathbf{A}_\epsilon + \frac{i}{2\kappa} (\Psi_\epsilon^* \nabla \Psi_\epsilon - \Psi_\epsilon \nabla \Psi_\epsilon^*) = \mathbf{0}, \quad \text{in } \Omega_{S_m T}, \quad m = 1, 2, \quad (5.3)$$

$$\frac{\partial \mathbf{A}_\epsilon}{\partial t} - \nabla^2 \mathbf{A}_\epsilon + |\Psi_\epsilon|^2 \mathbf{A}_\epsilon + \frac{i}{2\kappa} (\Psi_\epsilon^* \nabla \Psi_\epsilon - \Psi_\epsilon \nabla \Psi_\epsilon^*) = \mathbf{0}, \quad \text{in } \Omega_{NT}, \quad (5.4)$$

with boundary conditions,

$$\frac{\partial \Psi_\epsilon}{\partial n} = 0, \quad \mathbf{A}_\epsilon \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (5.5)$$

$$\operatorname{curl} \mathbf{A}_\epsilon \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \partial\Omega \times (0, T), \quad (5.6)$$

interface conditions,

$$[\mathbf{A}_\epsilon]_{\Gamma_{mT}} = \mathbf{0}, \quad [\Psi_\epsilon]_{\Gamma_{mT}} = 0, \quad \text{for } m = 1, 2, \quad (5.7)$$

$$[\nabla \cdot \mathbf{A}_\epsilon]_{\Gamma_{mT}} = 0, \quad \text{for } m = 1, 2, \quad (5.8)$$

$$[(\nabla - i\kappa \mathbf{A}_\epsilon) \Psi_\epsilon]_{\Gamma_{mT}} = \mathbf{0}, \quad \text{for } m = 1, 2, \quad (5.9)$$

$$[\operatorname{curl} \mathbf{A}_\epsilon \times \mathbf{n}]_{\Gamma_{mT}} = \mathbf{0}, \quad \text{for } m = 1, 2, \quad (5.10)$$

and initial conditions,

$$\Psi_\epsilon(\mathbf{x}, 0) = \Psi_\epsilon^0(\mathbf{x}),$$

$$\mathbf{A}_\epsilon(\mathbf{x}, 0) = \mathbf{A}_\epsilon^0(\mathbf{x}),$$

where $\Omega_T = \Omega \times (0, T)$, $\Omega_{S_m T} = \Omega_{S_m} \times (0, T)$, for $m = 1, 2$, $\Omega_{NT} = \Omega_N \times (0, T)$, and the interface is given by $\Gamma_{mT} = \Gamma_m \times (0, T)$ for $m = 1, 2$.

The same as in the §3, one can show that the leading order equations are:

$$\left(\frac{\partial \Psi}{\partial t} - i\kappa \Psi \nabla \cdot \mathbf{A} \right) - (\nabla - i\kappa \mathbf{A})^2 \Psi = \Psi(1 - |\Psi|^2), \quad \text{in } \Omega_{S_m T}^{(0)}, \quad m = 1, 2, \quad (5.11)$$

$$\frac{\partial \mathbf{A}}{\partial t} + \operatorname{curl}^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) + |\Psi|^2 \mathbf{A} = -\frac{i}{2\kappa} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*), \quad \text{in } \Omega_{S_m T}^{(0)}, \quad m = 1, 2, \quad (5.12)$$

with interface conditions,

$$[\mathbf{A}]_{\{z=0\} \times (0, T)} = \mathbf{0}, \quad [\Psi]_{\{z=0\} \times (0, T)} = 0, \quad (5.13)$$

$$[\nabla \cdot \mathbf{A}]_{\{z=0\} \times (0, T)} = 0, \quad (5.14)$$

$$\left[\left(\frac{\partial}{\partial z} - i\kappa a_3 \right) \Psi \right]_{\{z=0\} \times (0, T)} = 0, \quad (5.15)$$

$$[\operatorname{curl} \mathbf{A} \times \mathbf{k}]_{\{z=0\} \times (0, T)} = \mathbf{0}, \quad (5.16)$$

boundary conditions,

$$\frac{\partial \Psi}{\partial n} = 0, \quad \mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (5.17)$$

$$\operatorname{curl} \mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \partial\Omega \times (0, T), \quad (5.18)$$

and initial conditions

$$\Psi(\mathbf{x}, 0) = \Psi^0(\mathbf{x}),$$

$$\mathbf{A}(\mathbf{x}, 0) = \mathbf{A}^0(\mathbf{x}).$$

Let $(\Psi_\epsilon, \mathbf{A}_\epsilon)$ denote the solution of (5.1)–(5.4), we first quote the result from Hoffmann *et al.* [4]:

Lemma 4 *Assuming smooth conditions for the data:*

$$\partial D \in \mathcal{C}^2, \quad \Psi_\epsilon^0 \in \mathcal{H}^1(\Omega_N) \cap \mathcal{L}^\infty(\Omega_N), \quad \mathbf{A}_\epsilon^0 \in \mathbf{H}^1(\Omega_N),$$

$$\Psi_\epsilon^0 \in \mathcal{H}^1(\Omega_{S_m}) \cap \mathcal{L}^\infty(\Omega_{S_m}), \mathbf{A}_\epsilon^0 \in \mathbf{H}^1(\Omega_{S_m}), \quad m = 1, 2,$$

there exists a unique solution to (5.1)–(5.4) such that

$$\mathbf{A}_\epsilon \in H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^2(\Omega)),$$

$$\Psi_\epsilon \in L^2(0, T; \mathcal{H}^2(\Omega)) \cap H^1(0, T; \mathcal{L}^2(\Omega)) \cap \mathcal{L}^\infty(\Omega_T),$$

$$\|\Psi_\epsilon\|_{\mathcal{L}^\infty(\Omega_T)} \leq \max \{1, \|\Psi^0\|_{\mathcal{L}^\infty(\Omega)}\}.$$

By the weak compactness, this implies:

Proposition 3 *There exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$ such that*

$$\mathbf{A}_{\epsilon_k} \rightharpoonup \bar{\mathbf{A}} \quad \text{in } H^1(0, T; \mathbf{L}^2(\Omega)), \quad \text{and} \quad L^2(0, T; \mathbf{H}^2(\Omega)),$$

$$\Psi_{\epsilon_k} \rightharpoonup \bar{\Psi} \quad \text{in } H^1(0, T; \mathcal{L}^2(\Omega)), \quad \text{and} \quad L^2(0, T; \mathcal{H}^2(\Omega)).$$

By the compact imbedding $\mathbf{H}^1(0, T; \mathbf{L}^2(\Omega)) \hookrightarrow \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))$, we get:

Corollary 3 *There exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$ such that*

$$\mathbf{A}_{\epsilon_k} \rightarrow \bar{\mathbf{A}} \quad \text{in } \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))$$

$$\Psi_{\epsilon_k} \rightarrow \bar{\Psi} \quad \text{in } \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega)).$$

Similar to the steady state case, we may show that the limit is the solution of the original time-dependent GL equations (with no middle normal layer present). Based on the uniqueness of the solution for the time-dependent equations, we see that the whole sequence actually converges to the same limit.

For the time dependent case, we have the following result on order of convergence:

Theorem 3 *Assuming the same smooth conditions for the data as in Lemma 4. Then for small enough ϵ , there exists a constant $c > 0$, independent of ϵ , such that*

$$\|\Psi_\epsilon - \bar{\Psi}\|_{\mathcal{L}^2(0, T; \mathcal{H}^1(\Omega))} + \|\mathbf{A}_\epsilon - \bar{\mathbf{A}}\|_{L^2(0, T; H^1(\Omega))} \leq c\epsilon.$$

Proof See Appendix A. □

The order of convergence can be also observed in the numerical experiments presented in the next section.

6 The high-kappa, high-field limit

Many high- T_c superconductors have exceedingly large values of the Ginzburg-Landau parameter κ . In this regime, much of the discussion we have already examined can be further simplified, similar to the work of Chapman *et al.* [9] and Du & Gray [11].

To give a simple illustration, let us follow the discussion of Du & Gray [11] and choose the length scale to be ξ , the coherence length, and assume that the penetration depth λ (thus κ) is large with respect to the size of the superconducting sample. Let $\mathbf{H}_{\text{ext}} = \kappa \mathbf{H}_0$, where $\mathbf{H}_0$ is independent of κ , and define

$$\nabla \cdot \mathbf{B}_0 = 0 \quad \text{in } \Omega, \quad (6.1)$$

$$\text{curl } \mathbf{B}_0 = \mathbf{H}_0 \quad \text{in } \mathbb{R}^3, \quad (6.2)$$

then it is not difficult to show that to leading order, the time-dependent Ginzburg-Landau equations for the SNS junctions for the large κ limit can be written as:

$$\begin{aligned} \frac{\partial \Psi_{\epsilon,0}}{\partial t} - \frac{i}{\sigma} (\nabla \cdot \mathbf{A}_{\epsilon,0}) \Psi_{\epsilon,0} + (i\nabla + \mathbf{A}_{\epsilon,0} + \mathbf{B}_0)^2 \Psi_{\epsilon,0} \\ = \Psi_{\epsilon,0} (1 - |\Psi_{\epsilon,0}|^2), \quad \text{in } \Omega_{S_m T}, \quad m = 1, 2, \end{aligned} \quad (6.3)$$

$$\frac{\partial \Psi_{\epsilon,0}}{\partial t} - \frac{i}{\sigma} (\nabla \cdot \mathbf{A}_{\epsilon,0}) \Psi_{\epsilon,0} + \bar{m} (i\nabla + \mathbf{A}_{\epsilon,0} + \mathbf{B}_0)^2 \Psi_{\epsilon,0} = \alpha \Psi_{\epsilon,0} \quad \text{in } \Omega_{NT}, \quad (6.4)$$

$$\sigma \frac{\partial \mathbf{A}_{\epsilon,0}}{\partial t} - \nabla^2 \mathbf{A}_{\epsilon,0} = \mathbf{0} \quad \text{in } \Omega_{NT} \cup \Omega_{S_m T}, \quad m = 1, 2, \quad (6.5)$$

with boundary conditions,

$$(i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (6.6)$$

$$\text{curl} \mathbf{A}_{\epsilon,0} \times \mathbf{n} = \mathbf{0}, \quad \mathbf{A}_{\epsilon,0} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (6.7)$$

interface conditions,

$$[\Psi_{\epsilon,0}]_{\Gamma_{mT}} = 0, \quad [\mathbf{A}_{\epsilon,0}]_{\Gamma_{mT}} = \mathbf{0}, \quad \text{for } m = 1, 2, \quad (6.8)$$

$$(i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n}|_{\Gamma_{S_m T}} = \bar{m} (i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n}|_{\Gamma_{NT}}, \quad \text{for } m = 1, 2, \quad (6.9)$$

$$[\text{curl } \mathbf{A}_{\epsilon,0} \times \mathbf{n}]_{\Gamma_{nT}} = \mathbf{0}, \quad \text{for } m = 1, 2, \quad (6.10)$$

and initial condition,

$$\Psi_{\epsilon,0}(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \quad (6.11)$$

$$\mathbf{A}_{\epsilon,0}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}). \quad (6.12)$$

Here, σ is a material parameter that results from non-dimensionalization and $\bar{m} = m_S/m_N$, where m_S represents the mass of a superconducting charge carrier and m_N represents the mass of a non-superconducting charge carrier.

Note that in the special case when $\mathbf{A}_{\epsilon,0}(\mathbf{x}, 0) = \mathbf{0}$, the high-kappa equations (6.3)–(6.12) reduce to equations for $\Psi_{\epsilon,0}$ only:

$$\frac{\partial \Psi_{\epsilon,0}}{\partial t} + (i\nabla + \mathbf{B}_0)^2 \Psi_{\epsilon,0} = \Psi_{\epsilon,0} (1 - |\Psi_{\epsilon,0}|^2), \quad \text{in } \Omega_{S_m T}, \quad m = 1, 2,$$

$$\frac{\partial \Psi_{\epsilon,0}}{\partial t} + \bar{m} (i\nabla + \mathbf{B}_0)^2 \Psi_{\epsilon,0} = \alpha \Psi_{\epsilon,0} \quad \text{in } \Omega_{NT},$$

with boundary conditions,

$$(i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T),$$

interface conditions,

$$\begin{aligned} [\Psi_{\epsilon,0}]_{\Gamma_{mT}} &= 0, \quad \text{for } m=1, 2, \\ (i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n}|_{\Gamma_{SmT}} &= \bar{m} (i\nabla + \mathbf{B}_0) \Psi_{\epsilon,0} \cdot \mathbf{n}|_{\Gamma_{NT}}, \quad \text{for } m=1, 2, \end{aligned}$$

and initial condition,

$$\Psi_{\epsilon,0}(\mathbf{x}, 0) = \psi_0(\mathbf{x}).$$

If we examine the limit of the above equations as the normal layer thickness parameter $\epsilon \rightarrow 0$, we get the leading order equation with respect to ϵ ,

$$\frac{\partial \Psi_0}{\partial t} + (i\nabla + \mathbf{B}_0)^2 \Psi_0 = \Psi_0 (1 - |\Psi_0|^2), \quad \text{in } \Omega_{S_m^{(0)}T}, \quad m = 1, 2,$$

with boundary conditions,

$$(i\nabla + \mathbf{B}_0) \Psi_0 \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T),$$

interface conditions,

$$\begin{aligned} [\Psi_0]_{\{z=0\} \times (0, T)} &= 0, \\ \left[\left(i \frac{\partial}{\partial z} + b_{0,3} \right) \Psi_0 \right]_{\{z=0\} \times (0, T)} &= 0, \end{aligned}$$

and the next order correction with respect to ϵ is given by:

$$\begin{aligned} \frac{\partial \Psi_1}{\partial t} + (i\nabla + \mathbf{B}_0)^2 \Psi_1 &= \Psi_1 (1 - |\Psi_0|^2) \\ &\quad - (\Psi_1^* \Psi_0 + \Psi_1 \Psi_0^*) \Psi_0, \quad \text{in } \Omega_{S_m^{(0)}T}, \quad m = 1, 2, \end{aligned}$$

with boundary conditions,

$$(i\nabla + \mathbf{B}_0) \Psi_1 \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T),$$

and interface conditions,

$$\begin{aligned} [\Psi_1]_{\{z=0\} \times (0, T)} &= 0, \\ \left[\frac{\partial \Psi_1}{\partial z} \right]_{\{z=0\} \times (0, T)} &= \frac{2d_0}{\bar{m}} (\alpha + 1) \Psi_0(0, t) - \frac{2d_0}{\bar{m}} \Psi_0(0, t) |\Psi_0(0, t)|^2 \\ &\quad + (\bar{m} - 1) (i\nabla + \mathbf{B}_0(0))^2 \Psi_0(0, t). \end{aligned}$$

Notice that the above simplification is only meaningful when we consider the limit where, first, $\kappa \rightarrow \infty$, then $\epsilon \rightarrow 0$. In the ξ length scale, the leading order equation is the same if we take $\epsilon \rightarrow 0$ first, then $\kappa \rightarrow \infty$. The normal layer has negligible influence on the junction. Naturally, one can also consider other limiting cases, even cases where the magnitude of ϵ (as well as α and the diameter of the cross section D) may change with κ or *vice versa*. We will leave this to future works.

7 Numerical results

In this section we present the finite difference equations for a one-dimensional SNS junction in a simplified setting and use these equations to verify the order of convergence as well as the validity of the higher order corrections. We also demonstrate how the solution varies as the external applied field, $\mathbf{H}_{ext}$, changes and as the material parameter for the normal layer, α , changes.

7.1 Finite difference equations

We denote the solution of the SNS Junction by $\Psi_\epsilon = \Psi_{re} + i\Psi_{im}$, $\mathbf{A}_\epsilon = (a_1, a_2, a_3)$, and $\mathbf{H}_{ext} = (0, 0, h_3)$, where Ψ_ϵ and $\mathbf{A}_\epsilon$ depend only on x and $\mathbf{H}_{ext}$ is constant. If the initial conditions have the form

$$\Psi_\epsilon(x, 0) = \Psi_{re}(x), \quad (7.1)$$

$$\mathbf{A}_\epsilon(x, 0) = (0, a_2(x), 0), \quad (7.2)$$

then by uniqueness, the solution of the SNS junction will have the form

$$\Psi_\epsilon(x, t) = \Psi_{re}(x, t), \quad (7.3)$$

$$\mathbf{A}_\epsilon(x, t) = (0, a_2(x, t), 0), \quad (7.4)$$

where $\Psi = \Psi_{re}$ and $a = a_2$ satisfy

$$\frac{\partial \Psi}{\partial t} - \frac{\partial^2 \Psi}{\partial x^2} + \kappa^2 a^2 \Psi = \Psi(1 - \Psi^2), \quad \text{in } \Omega_{S_m T} \quad m = 1, 2, \quad (7.5)$$

$$\frac{\partial \Psi}{\partial t} - \frac{\partial^2 \Psi}{\partial x^2} + \kappa^2 a^2 \Psi = \alpha \Psi, \quad \text{in } \Omega_{NT}, \quad (7.6)$$

$$\frac{\partial a}{\partial t} - \frac{\partial^2 a}{\partial x^2} + \Psi^2 a = 0, \quad \text{in } \Omega_{S_1 T} \cup \Omega_{NT} \cup \Omega_{S_2 T}, \quad (7.7)$$

with boundary conditions:

$$\frac{\partial \Psi}{\partial x} = 0, \quad \frac{\partial a}{\partial x} = -h_3 \quad \text{on } \partial\Omega \times (0, T),$$

and interface conditions as before. Here, the sample under consideration occupies $\Omega = (-L, L)$ with superconducting material in both $\Omega_{S_1} = (\epsilon, L)$ and $\Omega_{S_2} = (-L, -\epsilon)$ and with normal material in $\Omega_N = (-\epsilon, \epsilon)$. By partitioning Ω into equal subintervals of length Δx ($\{-L = x_0 < x_1 < \dots < x_n = L\}$), with $x_N = \epsilon$ and $x_M = -\epsilon$, $r = \Delta t / \Delta x^2$, and using standard difference techniques, we obtain the following difference scheme:

$$\begin{aligned} -r\Psi_{j+1} + (1 + 2r + \kappa^2 \Delta t a_j^2)\Psi_j - r\Psi_{j-1} \\ - \Delta t \Psi_j(1 - \Psi_j^2) = \Psi_j^{old}, \quad j \in [1, M] \cup [N, n-1], \end{aligned} \quad (7.8)$$

$$-r\Psi_{j+1} + (1 + 2r + \kappa^2 \Delta t a_j^2)\Psi_j - r\Psi_{j-1} - \alpha \Delta t \Psi_j = \Psi_j^{old}, \quad j \in [M+1, N+1], \quad (7.9)$$

$$-ra_{j+1} + (1 + 2r + \Delta t \Psi_j^2)a_j - ra_{j-1} = a_j^{old}, \quad j \in [1, n-1], \quad (7.10)$$

with boundary condition at $x = -L$:

$$(1 + 2r + \kappa^2 \Delta t a_0^2) \Psi_0 - 2r \Psi_1 - \Delta t \Psi_0 (1 - \Psi_0^2) = \Psi_0^{old} \quad (7.11)$$

$$(1 + 2r + \Delta t \Psi_0^2) a_0 - 2r a_1 - 2r \Delta x h_3 = a_0^{old} \quad (7.12)$$

and interface condition at $x = -\epsilon$:

$$\Psi_{M+1} - 2\Psi_M + \Psi_{M-1} = 0. \quad (7.13)$$

Here, $\Psi_j = \Psi(x_j, t)$ and $\Psi_j^{old} = \Psi(x_j, t - \Delta t)$. Note that similar boundary and interface conditions hold at $x = L$ and $x = \epsilon$.

Similarly, the difference approximation of the leading order equations is given by:

$$\begin{aligned} -r \Psi_{j+1} + (1 + 2r + \kappa^2 \Delta t a_j^2) \Psi_j - r \Psi_{j-1} \\ - \Delta t \Psi_j (1 - \Psi_j^2) = \Psi_j^{old}, \quad j \in [1, n-1], \end{aligned} \quad (7.14)$$

$$-r a_{j+1} + (1 + 2r + \Delta t \Psi_j^2) a_j - r a_{j-1} = a_j^{old}, \quad j \in [1, n-1], \quad (7.15)$$

with boundary condition at $x = -L$:

$$(1 + 2r + \kappa^2 \Delta t a_0^2) \Psi_0 - 2r \Psi_1 - \Delta t \Psi_0 (1 - \Psi_0^2) = \Psi_0^{old} \quad (7.16)$$

$$(1 + 2r + \Delta t \Psi_0^2) a_0 - 2r a_1 - 2r \Delta x h_3 = a_0^{old} \quad (7.17)$$

and similar boundary conditions hold at $x = L$.

7.2 Convergence of the difference equations

Note that in the finite difference equations for the SNS junction (7.8)–(7.13), the grid size, Δx must be smaller than the width of the normal layer. Since we are interested in simulating the effect of smaller and smaller ϵ , we must choose smaller and smaller grid sizes to resolve the solution in the middle layer. In Figure 2, the finite difference scheme for the SNS junction is tested for convergence with two different mesh sizes.

7.3 Convergence of the leading order equations

We first use the finite difference equations to verify that the convergence is order ϵ . Figure 3 shows the error between the leading order solution, $|\bar{\Psi}|$, and the solution of the SNS Junction, $|\Psi_\epsilon|$, with $\epsilon = 0.02$ and $\epsilon = 0.01$ respectively. By comparing the magnitude in both graphs, we see the convergence of the leading order terms is order ϵ . Computationally, it is much simpler to implement the finite difference approximation of the leading order solution and there is no severe restriction on the grid size due to the parameter ϵ .

7.4 The next order corrections

It is not difficult to obtain a better approximation by computing the next order correction since the correction involves solving the single linear system derived in §3.2 with coefficients depending on the leading order solution. Continuing the last experiment, Figure 4 shows the error in the steady-state solution for $Re(\Psi)$ when we add the correction term

FIGURE 2. $|\Psi_\epsilon|$ with grid $\Delta t = 10^{-3}$, $\Delta x = 5 \times 10^{-2}$ (top) and grid $\Delta t = 5 \times 10^{-4}$, $\Delta x = 2.5 \times 10^{-2}$ (bottom).

FIGURE 3. Error in the leading order approximation with $\epsilon = 0.02$ (top) and 0.01 (bottom).

to the leading order solution (i.e. the graphs show $Re(\Psi^{(0)}) + \epsilon Re(\Psi^{(1)}) - Re(\Psi_\epsilon)$). Note that by comparing the graphs for $\epsilon = 0.02$ and $\epsilon = 0.01$, the error is order ϵ^2 .

7.5 Solutions for a junction with various fields

With the simplified model, the computation becomes much easier for small values of ϵ , since there is no need to distribute a sufficient number of grid points in the middle normal

FIGURE 4. Error in the solutions with correction: $\epsilon = 0.02$ (left) and $\epsilon = 0.01$ (right). FIGURE 5. Leading order $|\Psi|$ with different applied fields ($H_{ext} = 0.5, 0.7$ and 0.9). FIGURE 6. Corrected $|\Psi|$ using $\epsilon = 0.01$ with different applied fields ($H_{ext} = 0.5, 0.7$ and 0.9).

layer in order to show how it affects the behavior of the solution. We now give some experiments in which we use the leading order equation and the next order correction to investigate the properties of the SNS junctions. For example, by changing the external field, $\mathbf{H}_{ext}$, one may obtain different steady-state solutions to the SNS problem.

Figures 5–7 compare both the leading order solutions and the corrected solutions in the steady state for a sample of length of $L = 5$, Ginzburg–Landau parameter $\kappa = 1$, normal layer coefficient $\alpha = 1$, with various external fields at 0.5, 0.7, and 0.9, respectively. In the computation, $\Delta x = 5 \times 10^{-3}$, $\Delta t = 10^{-3}$ and constant initial conditions were used. For the corrected solution, $\epsilon = 0.01$ in Figure 6, while $\epsilon = 0.02$ in Figure 7.

The solution of the SNS Junction also depends on the material parameter α . The leading order solution remains the same, and the dependence can be shown by examining the next order correction. Figure 8 shows the corrected solutions with $\epsilon = 0.02$, external field $\mathbf{H}_{ext} = 0.9$, and with $\alpha = 1, 2$ and 10 , respectively.

More numerical results concerning the various other factors that may influence the properties of the SNS junctions may be found in Remski [12].

FIGURE 7. Corrected $|\Psi|$ using $\epsilon = 0.02$ with different applied fields ($H_{ext} = 0.5, 0.7$ and 0.9). FIGURE 8. Corrected $|\Psi|$ for varying values of α (1, 2 and 10).

8 Conclusion

By deriving the leading order equations and the next order correction, we have obtained a simplified model for the study of the SNS junction. These derivations are supported by both rigorous convergence estimates and numerical experiments and they provide a basis for further rigorous analysis of the *d.c.* and *a.c.* Josephson effects in the junction [1]. Even though a symmetric junction was used in the derivation of the model in this paper, similar ideas and analysis remain valid for the non-symmetric Junctions as well. The geometry of the Junction device is receiving more and more attention in recent studies of the high- T_c superconductors and of order parameter symmetry. The model derived here should find its application in the numerical simulations of various junction devices.

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Appendix A Order of convergence

In this appendix, we show that the convergence is of order ϵ .

Using the identity $\text{curl}^2 u = \nabla(\nabla \cdot u) - \Delta u$, the weak form of the equations (5.1)–(5.4) for an SNS junction are:

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ \frac{\partial \Psi_{\epsilon}}{\partial t} \phi - i\kappa \nabla \cdot \mathbf{A}_{\epsilon} \Psi_{\epsilon} \phi + (\nabla - i\kappa \mathbf{A}_{\epsilon}) \Psi_{\epsilon} \cdot (\nabla + i\kappa \mathbf{A}_{\epsilon}) \phi \right\} d\Omega dt \\ &= \sum_1^2 \int_0^T \int_{\Omega_{S_m}} \Psi_{\epsilon} (1 - |\Psi_{\epsilon}|^2) \phi d\Omega dt - \alpha \int_0^t \int_{\Omega_N} \Psi_{\epsilon} \phi d\Omega dt, \quad (\text{A } 1) \end{aligned}$$

$$\begin{aligned} \int_0^T \int_{\Omega} \left\{ \frac{\partial \mathbf{A}_{\epsilon}}{\partial t} \cdot \eta + \nabla \mathbf{A}_{\epsilon} \cdot \nabla \eta + |\Psi_{\epsilon}|^2 \mathbf{A}_{\epsilon} \eta + \frac{i}{2\kappa} (\Psi_{\epsilon}^* \nabla \Psi_{\epsilon} - \Psi_{\epsilon} \nabla \Psi_{\epsilon}^*) \cdot \eta \right\} d\Omega dt \\ = \int_0^T \int_{\partial\Omega} (\mathbf{H}_{\text{ext}} \times \mathbf{n}) \cdot \eta dS dt, \quad (\text{A } 2) \end{aligned}$$

where $\phi \in \mathcal{L}^2(0, T; \mathcal{H}^1(\Omega))$, $\eta = (\eta_1, \eta_2, \eta_3)^T \in L^2(0, T; \mathbf{H}^1(\Omega))$.

Similarly, the weak form for the leading order equations are:

$$\begin{aligned} \int_0^T \int_{\Omega} \left\{ \frac{\partial \Psi}{\partial t} \phi - i\kappa \nabla \cdot \mathbf{A} \Psi \phi + (\nabla - i\kappa \mathbf{A}) \Psi \cdot (\nabla + i\kappa \mathbf{A}) \phi \right\} d\Omega dt \\ = \sum_1^2 \int_0^T \int_{\Omega_{S_m^{(0)}}} \Psi (1 - |\Psi|^2) \phi d\Omega dt, \quad (\text{A } 3) \end{aligned}$$

$$\begin{aligned} \int_0^T \int_{\Omega} \left\{ \frac{\partial \mathbf{A}}{\partial t} \cdot \eta + \nabla \mathbf{A} \cdot \nabla \eta + |\Psi|^2 \mathbf{A} \eta + \frac{i}{2\kappa} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \cdot \eta \right\} d\Omega dt \\ = \int_0^T \int_{\partial\Omega} (\mathbf{H}_{\text{ext}} \times \mathbf{n}) \cdot \eta dS dt. \quad (\text{A } 4) \end{aligned}$$

Subtracting (A 3) from (A 1) and (A 4) from (A 2), writing $\Psi_d = \Psi_{\epsilon} - \Psi$ and $\mathbf{A}_d = \mathbf{A}_{\epsilon} - \mathbf{A}$ and adding and subtracting appropriate terms we obtain the following.

$$\begin{aligned} \int_0^T \int_{\Omega} \left\{ \frac{\partial \Psi_d}{\partial t} \phi + (\nabla - i\kappa \mathbf{A}_{\epsilon}) \Psi_d \cdot (\nabla + i\kappa \mathbf{A}_{\epsilon}) \phi + i\kappa (\nabla - i\kappa \mathbf{A}) \Psi_{\epsilon} \mathbf{A}_d \phi \right. \\ \left. - i\kappa (\nabla - i\kappa \mathbf{A}_{\epsilon}) \phi \mathbf{A}_d \Psi_{\epsilon} - i\kappa (\Psi_{\epsilon} \nabla \cdot \mathbf{A}_d + \Psi_d \nabla \cdot \mathbf{A}) \phi \right\} d\Omega dt \\ = \sum_1^2 \int_0^T \int_{\Omega_{S_m}} \{ \Psi_d - \Psi_{\epsilon}^3 + \Psi^3 \} \phi d\Omega dt + \int_0^t \int_{\Omega_N} \{ (-\alpha - 1) \Psi_{\epsilon} + \Psi_{\epsilon}^3 \} \phi d\Omega dt \quad (\text{A } 5) \end{aligned}$$

$$\begin{aligned} \int_0^T \int_{\Omega} \left\{ \frac{\partial \mathbf{A}_d}{\partial t} \cdot \eta + \nabla \mathbf{A}_d \cdot \nabla \eta + (|\Psi_{\epsilon}|^2 - |\Psi|^2) \mathbf{A}_{\epsilon} \eta + |\Psi|^2 \mathbf{A}_d \cdot \eta \right. \\ \left. + (\Psi_{\epsilon}^* \nabla \Psi_d + \Psi_d^* \nabla \Psi - \Psi_{\epsilon} \nabla \Psi_d^* - \Psi_d \nabla \Psi^*) \right\} d\Omega dt = 0. \quad (\text{A } 6) \end{aligned}$$

We now let $\phi = \chi_{(0,t)} \Psi_d^*$ in (A 5), where $\chi_{(0,t)}$ is the characteristic function for the interval $(0, t)$. Then (A 5) can be written as:

$$\begin{aligned} \int_0^t \int_{\Omega} \frac{\partial \Psi_d}{\partial t} \Psi_d^* d\Omega dt + \int_0^t \int_{\Omega} (\nabla - i\kappa \mathbf{A}_{\epsilon}) \Psi_d (\nabla + i\kappa \mathbf{A}_{\epsilon}) \Psi_d^* d\Omega dt \\ = -i\kappa \int_0^t \int_{\Omega} (\nabla - i\kappa \mathbf{A}) \Psi \cdot \mathbf{A}_d \Psi_d^* d\Omega dt + i\kappa \int_0^t \int_{\Omega} (\nabla - i\kappa) \Psi_d^* \cdot \mathbf{A}_d \Psi d\Omega dt \\ + i\kappa \int_0^t \int_{\Omega} (\Psi_{\epsilon} \nabla \cdot \mathbf{A}_d \Psi_d^* + |\Psi_d|^2 \nabla \mathbf{A}) d\Omega dt \\ + \sum_{m=1}^2 \int_0^t \int_{\Omega_{S_m}} \{ |\Psi_d|^2 + \Psi_d^* (\Psi^3 - \Psi_{\epsilon}^3) \} d\Omega dt \\ + \int_0^t \int_{\Omega_N} \{ (-\alpha - 1) \Psi_{\epsilon} + \Psi_{\epsilon}^3 \} \Psi_d^* d\Omega dt. \quad (\text{A } 7) \end{aligned}$$

Labelling the integrals in (A 7) by $I_1, I_2, \dots, I_7$, we now estimate each integral individually. Since $\Psi_d(x, 0) = 0$ we have:

$$\begin{aligned}
 I_1 &= \int_0^t \int_{\Omega} \frac{\partial \Psi_d}{\partial t} \Psi_d^* d\Omega dt = \frac{1}{2} \int_{\Omega} |\Psi_d|^2 d\Omega dt. \\
 I_2 &= \int_0^t \int_{\Omega} (\nabla - i\kappa \mathbf{A}_{\epsilon}) \Psi_d \cdot (\nabla + i\kappa \mathbf{A}_{\epsilon}) \Psi_d^* d\Omega dt \\
 &= \int_0^t \int_{\Omega} \left| \nabla \Psi_d - i\kappa \mathbf{A}_{\epsilon} \Psi_d \right|^2 d\Omega dt \\
 &\geq \int_0^t \int_{\Omega} \left| \nabla \Psi_d - \kappa |\mathbf{A}_{\epsilon} \Psi_d| \right|^2 d\Omega dt.
 \end{aligned} \tag{A 8}$$

By Hölder's inequality:

$$2\kappa \int_0^t \int_{\Omega} |\nabla \Psi_d| |\mathbf{A}_{\epsilon} \Psi_d| d\Omega dt \leq \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt + 2\kappa^2 \int_0^t \int_{\Omega} |\mathbf{A}_{\epsilon} \Psi_d|^2 d\Omega dt$$

Substituting this last expression into (A 8):

$$\begin{aligned}
 I_2 &\geq \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt - \kappa^2 \int_0^t \int_{\Omega} |\mathbf{A}_{\epsilon} \Psi_d|^2 d\Omega dt \\
 &\geq \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt - \kappa^2 \int_0^t \|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} |\Psi_d|^2 d\Omega dt. \\
 |\operatorname{Re}(I_3)| &= \left| \kappa \int_0^t \int_{\Omega} \nabla \Psi \cdot \mathbf{A}_d \Psi_d^* - i\kappa \mathbf{A} \Psi \cdot \mathbf{A}_d \Psi_d^* d\Omega dt \right| \\
 &\leq \int_0^t \|\nabla \Psi\|_{\mathcal{L}^4(\Omega)} \|\mathbf{A}_d\|_{\mathcal{L}^4(\Omega)} \|\Psi_d\|_{\mathcal{L}^2(\Omega)} dt \\
 &\quad + \frac{\kappa^2}{2} \int_0^t \int_{\Omega} |\mathbf{A} \cdot \mathbf{A}_d|^2 d\Omega dt + \frac{\kappa^2}{2} \int_0^t \int_{\Omega} |\Psi \Psi_d^*|^2 d\Omega dt \\
 &\leq \rho \int_0^t \|\mathbf{A}_d\|_{L^4(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \|\nabla \Psi\|_{\mathcal{L}^4(\Omega)}^2 \|\Psi_d\|_{\mathcal{L}^2(\Omega)}^2 dt \\
 &\quad + \frac{\kappa^2}{2} \int_0^t \|\mathbf{A}\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt + \frac{\kappa^2}{2} \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt.
 \end{aligned}$$

The last inequality holds for any $\rho > 0$ and c is a constant. Using the imbedding $\mathbf{A}_d \in H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $\nabla \Psi \in \mathcal{H}^1(\Omega) \hookrightarrow \mathcal{L}^4(\Omega)$, we have that for any $\rho > 0$:

$$\begin{aligned}
 |\operatorname{Re}(I_3)| &\leq \rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \|\Psi\|_{\mathcal{H}^2(\Omega)}^2 \int_{\Omega} |\Psi_d|^2 d\Omega dt \\
 &\quad + \frac{\kappa^2}{2} \int_0^t \|\mathbf{A}\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt + \frac{\kappa^2}{2} \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt. \\
 |\operatorname{Re}(I_4)| &\leq \frac{1}{16} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt + c \int_0^t \int_{\Omega} |\mathbf{A}_d \Psi|^2 d\Omega dt \\
 &\quad + c \int_0^t \int_{\Omega} |\mathbf{A}_{\epsilon} \cdot \mathbf{A}_d|^2 d\Omega dt + c \int_0^t \int_{\Omega} |\Psi_d^* \Psi|^2 d\Omega dt \\
 &\leq \frac{1}{16} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt + c \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt + c \int_0^t \|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt.
 \end{aligned}$$

As in the estimate for I_3 for all positive ρ :

$$\begin{aligned}
|\operatorname{Re}(I_5)| &= \left| \operatorname{Re} \left\{ i\kappa \int_0^t \int_{\Omega} \Psi_{\epsilon} \nabla \cdot \mathbf{A}_d \Psi_d^* + |\Psi_d|^2 \nabla \cdot \mathbf{A} \right\} d\Omega dt \right| \\
&\leq \kappa \int_0^t \int_{\Omega} |\Psi_{\epsilon} \nabla \cdot \mathbf{A}_d \Psi_d^*| d\Omega dt \\
&\leq \rho \int_0^t \int_{\Omega} |\nabla \cdot \mathbf{A}_d|^2 d\Omega dt + \frac{c}{\rho} \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt \\
&\leq \rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt.
\end{aligned}$$

Clearly, from the $L^\infty(\Omega_T)$ estimate on Ψ , Ψ_{ϵ} :

$$\begin{aligned}
|\operatorname{Re}(I_6)| &\leq c \int_0^t \int_{\Omega} |\Psi_d|^2 d\Omega dt, \\
|\operatorname{Re}(I_7)| &\leq c \int_0^t \int_{\Omega_0} |\Psi_d| d\Omega dt.
\end{aligned}$$

Now, we estimate I_7 by letting

$$f(z) = \int_D |\Psi_d|^2 d\Omega dy.$$

Then by Hölder's inequality, and noting that $\mathcal{W}^{1,3/2}(-1,1) \hookrightarrow \mathcal{L}^\infty(-1,1)$, we have

$$\begin{aligned}
\|f\|_{\mathcal{L}^\infty(-1,1)}^{3/2} &\leq c \|f\|_{\mathcal{W}^{1,3/2}(-1,1)}^{3/2} \\
&\leq c \int_{-1}^1 \int_D |\Psi_d|^3 d\Omega dy dz + c' \int_{-1}^1 \int_D |\Psi_d \nabla \Psi_d|^{3/2} d\Omega dy dz \\
&\leq c \|\Psi_d\|_{\mathcal{L}^3(\Omega)}^3 + c' \|\Psi_d\|_{\mathcal{L}^6(\Omega)}^{3/2} \|\nabla \Psi_d\|_{\mathcal{L}^2(\Omega)}^{3/2} \\
&\leq c \|\Psi_d\|_{\mathcal{L}^6(\Omega)}^3 + c' \|\nabla \Psi_d\|_{\mathcal{L}^2(\Omega)}^3 \leq c \|\Psi_d\|_{\mathcal{H}^1(\Omega)}^3,
\end{aligned} \tag{A 9}$$

where the last inequality follows from the imbedding $\mathcal{H}^1(\Omega) \hookrightarrow \mathcal{L}^6(\Omega)$. So, we have

$$\begin{aligned}
|\operatorname{Re}(I_7)| &\leq c \sqrt{\epsilon} \int_0^t \left(\int_{-\epsilon}^{\epsilon} f dz \right)^{1/2} dt \leq c\epsilon \int_0^t \|f\|_{\mathcal{L}^\infty(-1,1)}^{1/2} dt \\
&\leq c\epsilon \int_0^t \|f\|_{\mathcal{W}^{1,3/2}(-1,1)}^{1/2} dt \leq c\epsilon \int_0^t \|\Psi_d\|_{\mathcal{H}^1(\Omega)} dt \leq c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,t;\mathcal{H}^1(\Omega))}.
\end{aligned} \tag{A 10}$$

Putting these estimates together, we obtain the following:

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} |\Psi_d|^2 d\Omega + \frac{7}{16} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt \leq c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,t;\mathcal{H}^1(\Omega))} \\
&\quad + 3\rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \left(\|\Psi\|_{\mathcal{H}^2(\Omega)}^2 + \|\mathbf{A}_{\epsilon}\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} |\Psi_d|^2 d\Omega dt \\
&\quad + c \int_0^t \left(\|\mathbf{A}\|_{L^\infty(\Omega)}^2 + \|\mathbf{A}_{\epsilon}\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt.
\end{aligned} \tag{A 11}$$

Similarly, letting $\eta = \chi_{(0,t)} \mathbf{A}_d$ in (A 6) we have:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{A}_d|^2 d\Omega + \int_0^t \int_{\Omega} |\nabla \mathbf{A}_d|^2 d\Omega dt + \int_0^t \int_{\Omega} |\Psi|^2 |\mathbf{A}_d|^2 d\Omega dt \\ &= \int_0^t \int_{\Omega} (|\Psi_{\epsilon}|^2 - |\Psi|^2) \mathbf{A}_{\epsilon} \cdot \mathbf{A}_d d\Omega dt \\ & \quad + \frac{i}{2\kappa} \int_0^t \int_{\Omega} (\Psi_{\epsilon}^* \nabla \Psi_d + \Psi_d^* \nabla \Psi - \Psi_{\epsilon} \nabla \Psi_d^* - \Psi_d \nabla \Psi^*) \cdot \mathbf{A}_d d\Omega dt. \end{aligned}$$

Labeling the integrals on the right hand side above J_1 and J_2 , we have,

$$\begin{aligned} |\operatorname{Re}(J_1)| &\leq \left| \int_0^t \int_{\Omega} (\Psi_{\epsilon} \Psi_d^* + \Psi^* \Psi_d) \mathbf{A}_{\epsilon} \mathbf{A}_d d\Omega dt \right| \\ &\leq \int_0^t \|\Psi_d\|_{\mathcal{L}^2(\Omega)} \|\mathbf{A}_{\epsilon}\|_{L^4(\Omega)} \|\mathbf{A}_d\|_{L^4(\Omega)} dt \\ &\leq \rho \int_0^t \|\mathbf{A}_d\|_{L^4(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \|\mathbf{A}_{\epsilon}\|_{L^4(\Omega)}^2 \|\Psi_d\|_{\mathcal{L}^2(\Omega)}^2 dt \\ &\leq \rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} |\Psi_d|^2 d\Omega dt, \\ |\operatorname{Re}(J_2)| &\leq \frac{1}{2\kappa} \int_0^t \int_{\Omega} \{ |\nabla \Psi_d \cdot \mathbf{A}_d| + |\Psi_d \nabla \Psi \cdot \mathbf{A}_d| \} d\Omega dt \\ &\leq \frac{1}{16} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt + c \int_0^t \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt \\ & \quad + \rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{c}{\rho} \int_0^t \|\Psi\|_{\mathcal{H}^2(\Omega)}^2 \int_{\Omega} |\Psi_d|^2 d\Omega dt. \end{aligned}$$

Putting these estimates together gives:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{A}_d|^2 d\Omega + \int_0^t \int_{\Omega} |\nabla \mathbf{A}_d|^2 d\Omega dt \leq 2\rho \int_0^t \|\mathbf{A}_d\|_{H^1(\Omega)}^2 dt + \frac{1}{16} \int_0^t \int_{\Omega} |\nabla \Psi_d|^2 d\Omega dt \\ & \quad + \frac{c}{\rho} \int_0^t \left(\|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 + \|\Psi\|_{\mathcal{H}^2(\Omega)}^2 \right) \int_{\Omega} |\Psi_d|^2 d\Omega dt + c \int_0^t \int_{\Omega} |\mathbf{A}_d|^2 d\Omega dt. \end{aligned} \quad (\text{A } 12)$$

Now, adding (A 11) and (A 12) and choosing the parameter ρ sufficiently small, we obtain:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \{ |\Psi_d|^2 + |\mathbf{A}_d|^2 \} d\Omega + \frac{3}{8} \int_0^t \int_{\Omega} \{ |\nabla \Psi_d|^2 + |\nabla \mathbf{A}_d|^2 \} d\Omega dt \\ & \leq c \int_0^t h(s) \int_{\Omega} \{ |\Psi_d|^2 + |\mathbf{A}_d|^2 \} d\Omega ds + c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,t;\mathcal{H}^1(\Omega))}, \end{aligned} \quad (\text{A } 13)$$

where $h(t) = \sup \left(\|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 + \|\Psi\|_{\mathcal{H}^2(\Omega)}^2, \|\mathbf{A}\|_{L^{\infty}(\Omega)}^2 + \|\mathbf{A}_{\epsilon}\|_{L^{\infty}(\Omega)}^2 \right) \in L^1(0, T)$. Ignoring the gradient terms on the left-hand side of inequality (A 13) and applying Gronwall's inequality gives:

$$\begin{aligned} \int_{\Omega} \{ |\Psi_d|^2 + |\mathbf{A}_d|^2 \} d\Omega &\leq c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))} \left(1 + \int_0^t h(s) e^{H(t)-H(s)} ds \right) \\ &\leq c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))} \end{aligned} \quad (\text{A } 14)$$

where $H(t) = \int_0^t h(s)ds$. Now, by substituting this last expression into the right-hand side of (A 13) and integrating over $[0, T]$, we obtain:

$$\|\Psi_d\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}^2 + \|\mathbf{A}_d\|_{L^2(0,T;H^1(\Omega))}^2 \leq c\epsilon \|\Psi_d\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}$$

which gives the desired order of convergence.

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