Application of 2x2 and 4x4 matrices to the modeling of all nematic liquid crystal displays

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ABSTRACT

The 2x2 Jones matrix and 4x4 Mueller matrix methods are used as computational tools for LCD quiescent state optical mode design. Matrices of a general twisted nematic (GTN) LC cell and some basic optical components are given. The transmission or reflection of a GTN cell can be calculated easily by using these matrices. A parameter space approach of designing LCDs is described, by which many GTN optical modes are revealed with information on their transmittance/reflectance, dispersion and also cell gap tolerance. The 4x4 Mueller matrix is introduced both as an alternative to the Jones matrix calculation and an efficient tool for investigating GTN cell polarization transformation. A polarization transformation switching scheme is proposed to explain all GTN display operations. New bistable twisted nematic (BTN) modes are invented by considering horizontal switching of linear polarization preservation modes.

Keywords: Jones matrix, Mueller matrix, polarization transformation, LCD modeling.

1. INTRODUCTION

2x2 Jones calculus is a common tool for calculating the evolution of polarized light through linear optical elements. This method is also very useful for calculating quiescent state optical performance of a general twisted nematic (GTN)¹ LC cell. By knowing the Jones matrix of respective optical elements such as a GTN cell, polarizer, retarder and mirror, the transmission or reflection of a GTN LCD can be calculated easily. The final result is in terms of light intensity. While the Jones matrices and Jones vectors are in general complex, an alternative approach of calculating GTN LCD optics by the 4x4 Mueller calculus is completely real. In addition, the manipulating object, Stokes vectors, are measurable quantities. As all GTN LCDs operate by polarization manipulation. It is straightforward to design GTN LCD modes by Mueller calculus. No matter which matrix method you are using, the result is either a number or a vector. It takes a lot of time for trial and error before one can obtain a good performance optical mode. The parameter space (PS) approach of GTN optical mode design can help us to visualize the correct parameters for good modes at a glance. In general, a GTN cell optical performance is a function of four parameters, $T(\phi, d\Delta n, \alpha, \gamma)$, where ϕ is the twist angle, d is the cell gap thickness, Δn is the LC birefringence, α and γ are the front and rear polarizer angle respectively. In practice, very often ϕ and $d\Delta n$ are variables, while the polarizer angles are fixed. In this paper, the techniques of using the 2x2 Jones matrix and parameter space to design GTN LCDs are introduced. And then 4x4 Mueller matrix is used to develop equations that govern the polarization transformations. Then by applying these equations, it is shown that all GTN LCD optical mode operations can be explained by means of switching between the polarization transformation modes. New bistable twisted nematic (BTN) operation modes are also discovered by employing the polarization transformation switching concept.

2. 2x2 JONES MATRIX

The Jones matrix of a GTN cell and some basic optical elements are listed in table 1. By multiplying the matrices in correct order, the final Jones vector is obtained. The resulting light intensity is just the amplitude square of the final Jones vector. Eq. (1) gives the matrix multiplication order of a transmissive GTN LCD calculation. Eq. (2) is the calculation for a single polarizer reflective GTN LCD.

$$J'_T = \text{Pol}(\gamma) \bullet \text{LC}(\phi, d\Delta n) \bullet \text{Pol}(\alpha) \bullet J \quad (1)$$

$$J'_R = \text{Pol}(-\alpha) \bullet \text{Rot}(\phi) \bullet \text{LC}(\phi, d\Delta n) \bullet \text{Rot}(-\phi) \bullet \text{Mir} \bullet \text{LC}(\phi, d\Delta n) \bullet \text{Pol}(\alpha) \bullet J \quad (2)$$

Retardation films² or other optical elements can be added to the calculation if necessary. By using eqs.(1) and (2), a parameter space representation is developed. The $(\phi, d\Delta n)$ set is the most important parameter set. It completely determines the Jones matrix of a GTN LC cell.

Table 1. The Jones matrices of a GTN LC cell and associated optical elements.

Optical Element	Jones Matrix	Remarks
GTN LC cell	$LC = \begin{bmatrix} a + ib & -c + id \\ c + id & a - ib \end{bmatrix}$	$a = \cos\phi\cos\beta + (\phi/\beta)\sin\phi\sin\beta$ $b = -(\delta/\beta)\cos\phi\sin\beta$ $c = \sin\phi\cos\beta - (\phi/\beta)\cos\phi\sin\beta$
Polarizer	$Pol = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$	$d = -(\delta/\beta)\sin\phi\sin\beta$ $\delta = \pi d\Delta n/\lambda$
Retarder	$Ret = \begin{bmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{bmatrix}$	$\beta = (\delta^2 + \phi^2)^{1/2}$ $\phi = \text{twist angle}$ $\theta = \text{angle to the x-axis}$
Mirror	$Mir = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\Delta n = n_e - n_o$ $n_e = \text{extraordinary index}$ $n_o = \text{ordinary index}$
Rotator	$Rot = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$	

3. PARAMETER SPACE (PS) REPRESENTATION

By varying the retardation and twist angle using eq. (1), a parameter space contour plot¹ is generated. In Fig. 1, the front polarizer angle α is parallel to the input director, and the rear polarizer is cross to the front polarizer. Every successive line in the contour plot represents an increase of 0.1 in transmittance. One has to look for the modes with highest transmittance. Obviously, there are two very good modes. They are the TN1 and TN2 modes as depicted in Fig. 1. These modes are correspondingly the 90° TN first and second minimum of Gooch and Tarry³. The small dispersion and large cell gap margin of TN modes can be recognized by observing that the $T > 0.9$ area is extended in the retardation axis. This is explained by the fact that retardation actually includes the factor $d\Delta n/\lambda$, thus large retardation margin means small color dispersion and large cell gap tolerance. In Fig. 2, the PS contour plot of birefringence modes with $\alpha = 45^\circ$ is given. The output polarizer angle γ is set to be equal to $\phi + \alpha$. The zero twist electrically controlled birefringence (ECB)⁴ modes and supertwisted nematic (STN)⁵ modes are examples. The coloration of birefringence modes and small cell gap margin can be inferred by the short retardation extension with $T > 0.9$. The optical mode interference (OMI)^{6,7} mode is also shown in Fig. 1. It is designed to solve the color problem of STN displays. One can see that a rather large retardation tolerance is found for the OMI mode with parameter $(180^\circ, 0.56\mu\text{m})$. However the low optical efficiency is also evident by observing that it has only a transmittance of 0.5. In all plots, wavelength is assumed to be at 550nm. Retardation unit is in micrometer and twist angle unit is in degree.

When the PS representation is applied to single polarizer reflective displays, one can obtain Fig. 3 easily. The input polarizer angle is parallel to the input director. The normally black solutions with $(\phi, d\Delta n)$ equal to $(63^\circ, 0.19\mu\text{m})$ and $(190^\circ, 0.58\mu\text{m})$ are expected. They are named TN-ECB1⁸ and TN-ECB2 in Fig. 3. The normally white hybrid field effect (HFE)⁹ mode and reflective TN (RTN)¹⁰ mode can also be found on the same plot. By introducing a non-zero α , a new reflective STN (RSTN)¹¹ modes has been invented by Chen et al using the PS method.

Figure 1. PS contour plot with $\alpha=0^\circ$ and cross analyzer.

Figure 2. PS contour plot with $\alpha=45^\circ$ and $\gamma=\phi+\alpha$. STN displays lie on the $T>0.9$ band.

Figure 3. Single polarizer reflective modes with $\alpha=0^\circ$. The TNECB modes, HFE and RTN modes are shown.

Figure 4. CP solutions in PS representation are shown. The normally black CP states are gathered at the $R<0.1$ region.

Up to now we have demonstrated how some of the optical modes can be obtained on a PS contour plot without solving any equation. In the next section, the 4x4 Mueller matrix method is introduced, by which some very useful analytical equations for LCD optical mode design are derived.

4. 4x4 MUELLER MATRIX AND POLARIZATION TRANSFORMATION

All 2x2 Jones matrices can be transformed into corresponding 4x4 Mueller matrices¹². Eqs. (1) and (2) are still valid for Mueller matrix LCD modeling. The Mueller matrix of a GTN cell can be written as

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2 + b^2 - c^2 - d^2 & 2(bd - ac) & -2(ad + bc) \\ 0 & 2(ac + bd) & a^2 - b^2 - c^2 + d^2 & 2(ab - cd) \\ 0 & 2(ad - bc) & -2(ab + cd) & a^2 - b^2 + c^2 - d^2 \end{pmatrix}$$

The polarization transform by a GTN cell is studied through the equation $J'=M \cdot J$. In general, J' is the resulting Stokes vector representing an elliptically polarized light. However, under certain $(\phi, d\Delta n, \alpha)$ combination, one can have linear or circular polarized output¹³. There are two conditions under which the GTN cell can give linear outputs. They are called LP1 and LP2 solution as given in eqs. (3) and (4).

$$\text{LP1} \quad \delta^2 + \phi^2 = (N\pi)^2 \quad (3a)$$

$$\gamma = \phi + \alpha \quad (3b)$$

$$\text{LP2} \quad \frac{\phi}{\sqrt{\delta^2 + \phi^2}} \tan \sqrt{\delta^2 + \phi^2} = \tan 2\alpha \quad (4a)$$

$$\gamma = \phi - \alpha \quad (4b)$$

There also exist conditions for which circular outputs can be obtained, they are the CP solutions.

$$\text{CP} \quad \frac{\delta^2}{\delta^2 + \phi^2} \sin^2 \sqrt{\delta^2 + \phi^2} = \frac{1}{2} \quad (5a)$$

$$\tan 2\alpha = \frac{-\sqrt{\delta^2 + \phi^2}}{\phi} \cot \sqrt{\delta^2 + \phi^2} \quad (5b)$$

The solutions of eq.(3) (4) and (5) are given in Fig. 5. All common GTN modes are also labeled in appropriate positions. It can be observed that all those modes lie on one of these curves. It is by no means accidental. The reason behind is that for transmissive displays, one requires a linear to linear polarization preservation and for reflective displays, one requires a linear to circular transformation. In order to fulfill these requirements, all GTN modes have their field off states lie on these curves. By implementing a switching scheme between these curves, all GTN operations are explained.

Figure 5. The LP1(solid curves), LP2 with $\alpha=45^\circ$ (dashed curves) and CP (thick curves) solutions of a GTN LC cell. All existing GTN modes lie on one of these curves.

Table 2 lists the common GTN display modes and their switching principles under the polarization transformation switching scheme. The non-select state (ns) is the field off state and the design parameters can be analytically solved. The

select state (s) is the field on state. With the electric field on, the assumption of uniform tilt and twist of the GTN matrix is no longer valid. For non-homeotropic select states, the LP2 or CP mode select states are only 'effective' linear and circular states. They cannot be represented on the plot in an exact sense. However it is still beneficial conceptually to include these under a general operation scheme. The CP solution PS contour plot is given in Fig. 4. The $R < 0.1$ region is ideal for reflective display with normally black CP to H switching. The TN-ECB1, mixed mode TN (MTN)¹⁴ and self-compensated TN (SCTN)¹⁵ modes are designed within or near the $R < 0.1$ region. Optimized mode can be obtained by selecting the CP solution exactly on the curve in Fig. 5.

Optical modes	$(\phi, d\Delta n)$	ns $\rightarrow$ s	α	T/R
TN	$(90^\circ, 0.48\mu\text{m})$ $(90^\circ, 1.0\mu\text{m})$	LP1 $\rightarrow$ H	0°	T
STN	$(240^\circ, 0.82\mu\text{m})$	LP1 $\rightarrow$ LP2	45°	T
TN-ECB	$(63^\circ, 0.19\mu\text{m}), (190^\circ, 0.6\mu\text{m})$	CP $\rightarrow$ H	0°	R
MTN	$(90^\circ, 0.25\mu\text{m})$	CP $\rightarrow$ H	20°	R
SCTN	$(60^\circ, 0.35\mu\text{m})$	CP $\rightarrow$ H	30°	R
HFE	$(45^\circ, 0.54\mu\text{m})$	LP1 $\rightarrow$ CP	0°	R*
RTN	$(52^\circ, 0.52\mu\text{m})$	LP1 $\rightarrow$ CP	0°	R*
RSTN	$(240^\circ, 0.85\mu\text{m})$	LP1 $\rightarrow$ CP	6°	R*

Table 2. General LCD modes with their $(\phi, d\Delta n)$ and operating principles are listed. H represents a homeotropic state when high voltage is applied. T/R indicates whether the LCD works in transmittive or reflective mode. The R modes are normally black while the R* modes are normally white. The NB/NW state is reversed if a polarizing beam splitter (PBS) is used for single polarizer reflective cell.

5. SPECIAL APPLICATION TO BISTABLE NEMATIC MODES DESIGN

In section 4, all the operating modes from non-select state to select state work by reducing the effective retardation as applied voltage increases. For the H (homeotropic) state, in fact the retardation is reduced to zero. We can call these kinds of switching the vertical switching. However, there are also horizontal switching that keep the retardation value unchanged while only the twist angle change. Bistable twisted nematic (BTN) modes belong to this group. Bistable twisted nematic displays are optical modes that have two quasi-stable states that have their twist angles differ by 2π . Usually, the twist angle ϕ of a GTN cell is defined by the surface rubbing direction. However the same rubbing direction is also good for twist angle of $\phi + 2\pi$. By doping the LC cell deliberately to have an intermediate twist state between ϕ and $\phi + 2\pi$, it is possible to drive the LC cell between ϕ state and $\phi + 2\pi$ state. The first practical BTN¹⁶ display works on a 0 to 2π switching. Details of the driving scheme and mechanism can be found in references 17-18. Here, we would like to introduce the method for obtaining optimized BTN configurations. The following rules must be obeyed for optimized BTN modes.

1. It is a switching between LP modes;
2. It is a horizontal switching with 2π difference in twist;
3. The two quasi-stable states have their output polarization directions cross to each other.

Rules 1 and 3 requires a switching between LP1 and LP2 with $\alpha = 45^\circ$. Graphical solutions of transmittive BTN are shown in Fig. 6.

Figure 6. Two sets of transmittive BTN solutions are shown (o and Δ) for easy distinguish. They are LP1 to LP2 switching. The twist difference for each solution is 2π . The twist angles are not restricted to multiples of $\pi/2$ as reported before^{19,20}. Some of these modes show excellent brightness and contrast through the whole visible spectrum. Note that the graphical solution is restricted to be larger than -200° just for convenience. It is obvious that the solutions are symmetric about the zero-twist axis. That means sign interchange of the two twist is immaterial.

Numerical solutions are also obtained by solving eqs. (3) and (4) simultaneously under the above three conditions. By letting ϕ_1 be the twist angle of LP1 solution and $\phi_2 = \phi_1 + 2\pi$ be the twist angle of LP2 solution, the first six solutions with $d\Delta n$ less than $1\mu\text{m}$ are obtained and listed in table 3 below. The rear polarizer angle γ is assigned such that maximum contrast is obtained.

$d\Delta n / \mu\text{m}$	ϕ_1	ϕ_2	γ
0.273	-348.75	11.25	-33.75
0.399	-123.75	236.25	-78.75
0.522	56.25	416.25	101.25
0.733	-483.75	-123.75	11.25
0.765	-258.75	101.25	56.25
0.932	191.25	551.25	56.25

Table 3. The first six optimized transmissive BTN solutions are listed. λ is assumed to be at 550 nm.

6. CONCLUSIONS

The 2x2 Jones matrix calculus for calculation of both transmissive and reflective GTN displays is introduced. This method can give us fast result on all GTN displays transmission or reflection with normal incident light. The most important parameters are the $(\phi, d\Delta n)$ set. It uniquely determines the matrix of a GTN cell. By varying these two parameters, a parameter space is constructed. Contour plots on transmittance or reflectance is drawn. By examining the resultant PS contour plot, modes with good optical performance can be selected. The color dispersion characteristic and cell gap margin of GTN cells can also be inferred. While the PS method gives us graphical solution of GTN mode design, the 4x4 Mueller matrix method can help us to obtain exact design parameters. Polarization transformations of a GTN cell are investigated by the Mueller matrix method. A polarization transformation scheme to explain all GTN operation is attempted. Application of the linear polarization switching principle to BTN design has lead to new operation modes.

7. REFERENCES

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