Valuing employee reload options under time vesting requirement

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Upon the exercise of an employee stock option, the embedded reload provision entitles the holder to receive additional units of new options from the employer. The number of units of new options received is equal to the number of shares tendered as payment of strike and the new strike is set at the prevailing stock price. The reload provision may be subject to time vesting requirement, where the employee is prohibited from exercising the reload until the end of an initial vesting period. After each exercise, the new reload options are subject to the same time vesting constraint. In this paper, we present the linear complementarity formulation of the pricing models and construct efficient numerical algorithms that evaluate the market value of the employee reload options under time vesting requirement. Also, we explore the analytic properties of the price functions and optimal exercise policies of the employee reload options.

1 INTRODUCTION

Employee (executive) stock options commonly contain non-traditional features that are not found in other conventional options traded in the financial markets. Johnson and Tian's paper (2000) contains an interesting account of various types of exotic features in employee stock options. The reload provision in employee stock options has been gaining increasing popularity since its first launch in late 1980s. Suppose that at the exercise of an employee stock option prior to expiry, the employee pays the strike of the option
with his owned shares. The embedded reload feature entitles the employee to receive one new option for each share tendered for the strike payment, in addition to receiving one share for the option exercised. The new reload option has the same expiration date as the original option but with a new strike price set at the prevailing stock price. Let $X$ be the strike of the original option and $S$ be the prevailing stock price at the exercise moment. Upon exercise, the employee receives $1 - X/S$ units of shares and $X/S$ units of new reload option. Furthermore, the new option received may continue to have future reload rights, and the number of allowable reloads can be unlimited or finite. In addition, there may be vesting restrictions on exercising the reload. The performance vesting requires a minimum level of stock price appreciation before exercise is allowed. In time vesting requirement, the first reload can be exercised only after an initial vesting period (say, 6-month period). The new reload options received after each exercise are subject to the same vesting period. More details of the product nature of different types of reload options can be found in the report by Frederic W. Cook & Co. (1998).

It is important for firms to estimate the compensation costs associated with the granting of such reload options to their employees. The valuation of the cost to the employer seems to be quite difficult, as noted by the Financial Accounting Standards Board (see Statement of Financial Accounting Standards No. 123, 1995). Since the employee options are non-transferable and the employees cannot short sell the stocks to hedge the risk, the exercise decision cannot be preference free. Also, the wealth of the holder depends too highly on the fortune of the firm. Therefore, a risk averse holder would have higher tendency to exercise the options prematurely in order to guard against potential acute drop in stock price. Apparently, it seems impossible to derive a general preference free pricing method of finding the cost to the issuer of such reload options.

However, when the number of reloads is unlimited, Dybvig and Loewenstein (2003) proved by dominance argument that there exists a simple exercise policy - the holder should exercise whenever a new maxima of the stock price is realized. Such optimal policy is independent of the risk preferences of the holders. Though the utility value of the cashflows at exercise depends on the utility function adopted by the risk averse holder, but the exercise policy remains the same as that of a risk neutral holder. Since the employee reload option can be hedged by the issuer, the expected value of the cashflows discounted under the riskless interest rate should represent the market value of the option. Further, as the employer is aware of such optimal exercise policy
adopted by the holder, and he has no constraint on hedging, so the market value of the options also represents the cost to the employer granting such options. These arguments pave the way for the use of the Black-Scholes risk neutral pricing paradigm for valuation of the cost to the issuer of employee options with unlimited reloads.

Analytic properties of the pricing models for finding the market value of reload options have been analyzed in our earlier paper (Dai and Kwok, 2003). In this paper, we would like to extend the analysis of the pricing models of reload options with the inclusion of time vesting requirement. We develop efficient valuation algorithm for computing the option values and examine the behaviors of the price functions. Dybvig and Loewenstein (2003) argued that since the valuation problem with time vesting exhibits close proximity to that without time vesting, therefore as an approximation, the market value of the option is taken to be the cost to the employer. Several earlier papers have attempted to solve similar valuation problems of reload options. Hemmer et al. (1998) bypassed the complexities of valuing a time vesting infinite-reload option through approximating the time vesting requirement by an alternative restriction where exercise is allowed only at preset times. If the time interval between these allowable exercise instants is the same as the time vesting period, then the value of the option under the alternative restriction is always less than that under time vesting since the exercise at preset instants can only be sub-optimal. Huddart et al. (1999) developed binomial algorithms for finite-reload options and considered the impact of various types of exercise payoff functions. In the trinomial scheme developed by Dybvig and Loewenstein (2003) for valuation of infinite-reload options with time vesting, they append an extra index for tracking the vesting time lapsed from the last exercise instant.

The paper is organized as follows. In the next section, we discuss the linear complementarity formulation for the continuous pricing model for valuing reload options under time vesting requirement. We explore some analytic properties of the price functions. Once the continuous pricing models are well formulated, it becomes quite straightforward to construct efficient trinomial schemes for numerical valuation of the option value. In Section 3, we present the results of the numerical experiments that were designed to reveal some of the behaviors of the pricing functions. In particular, we explore the characterization of the optimal exercise policies of the reload options and the impact of the length of the vesting period on the price functions. The paper ends with conclusive remarks in the last section.
2 FORMULATION OF PRICING MODELS

We would like to formulate the continuous pricing models that evaluate the market value of reload options with time vesting requirement. We adopt the usual risk neutral pricing approach under the Black-Scholes framework. Under the risk neutral measure, the stochastic process of the stock price $S$ is assumed to follow the Geometric Brownian process

$$\frac{dS}{S} = (r - q) \, dt + \sigma \, dZ,$$  \hspace{1cm} (2.1)

where $r$ and $q$ are the constant riskless interest rate and dividend yield, respectively, $\sigma$ is the constant volatility, $dZ$ is the standard Wiener process and $t$ is the calendar time variable. Let $T$ denote the expiration date of the reload option and write $\tau = T - t$ as the time to expiry. For convenience, we take the original strike price $X$ of the reload option to be unity. Also, we let $\delta$ denote the length of the time vesting period.

Let $V_n(S, t, u, X)$ denote the price function of the employee option with $n$ reloads ($n$ can be infinite), where $u$ denotes the remaining time required to satisfy the time vesting requirement and $X$ is the strike price. Obviously, we have $0 \leq u \leq \delta$, and the (weak) path dependence nature of the time vesting feature is exhibited by the additional state variable $u$. In particular, once $u$ hits the value 0, the holder can exercise the reload at any time he wishes. Right after an exercise, $u$ is reassigned the value $\delta$. The price function $V_n(S, t, u; X)$ satisfies the linear homogeneity property when the strike price equals the stock price, that is

$$V_n(S, t, u; S) = SV_n(1, t, u; 1).$$  \hspace{1cm} (2.2)

Suppose there are $n$ reload rights outstanding, $n \geq 1$, the payoff to the holder upon exercise is given by (the strike price $X$ is set to be unity)

$$S - 1 + \frac{1}{S} V_{n-1}(S, t, \delta; S) = S - 1 + V_{n-1}(1, t, \delta; 1).$$  \hspace{1cm} (2.3)

We assume that the option received at the last reload is an American vanilla call option but subject to no early exercise privilege in the next $\delta$-time period. Only at the end of the time vesting period does this option become the usual American call option. We then have $V_0(S, t, 0) = C_A(S, t)$, where $C_A(S, t)$ is the price function of the American call with unit strike.
2.1 Vesting time variable

The price function $V_n(S, t, u)$ apparently contains an extra vesting time variable $u$, similar to the excursion time variable in Parisian options. The path dependence nature of the time vesting feature is reflected by the property that $u$ is set equal to the grant date and reset to zero upon each exercise. Fortunately, there exists a simple calculation procedure to obtain the solution $V_n(S, t, u), 0 < u \leq \delta$, once the solution $V_n(S, t + u, 0)$ is available. As there is no reload allowed inside the time interval $[t, t + u)$, the reload option behaves like a compound option. Over the interval $[t, t + u)$, the price function $V_n(S, t, u)$ satisfies the Black-Scholes equation without optimal stopping, subject to “known” payoff at $t + u$ as given by $V_n(S, t + u, 0)$. Let $\psi(S_{t+u}|S_t = S)$ denote the transition density function of the stock price at time $t + u$, conditional on the stock price at time $t$ equals $S$. It is straightforward to deduce the following relation:

$$V_n(S, t, u) = e^{-ru} \int_0^\infty \psi(S_{t+u}|S_t = S)V_n(S_{t+u}, t + u, 0) \, dS_{t+u},$$

(2.4)

where

$$\psi(S_{t+u}|S_t) = \frac{1}{\sigma\sqrt{2\pi u} S_{t+u}} \exp \left( -\frac{\left[ \ln \frac{S_{t+u}}{S_t} - \left( r - q - \frac{\sigma^2}{2} \right) u \right]^2}{2\sigma^2 u} \right).$$

(2.5)

This is like an integral transform on $V_n(S, t + u, 0)$ with the kernel function $e^{-ru}\psi(S_{t+u}|S_t)$. For notational convenience in later discussion, we define the integral transform operator $F_u^S$ by

$$V_n(S, t, u) = F_u^S[V_n(\tilde{S}, t + u, 0)] = \int_0^\infty e^{-ru}\psi(\tilde{S}|S)V_n(\tilde{S}, t + u, 0) \, d\tilde{S},$$

(2.6)

where $S_{t+u} = \tilde{S}$ and $S_t = S$. Accordingly, we may rewrite the exercise payoff in Eq. (2.3) as $S - 1 + F_u^S[V_{n-1}(\tilde{S}, t + \delta, 0)]_{S-1}$. Since the function $V_n(S, t, u), u > 0$, is related directly to $V_n(S, t, 0)$ by the relation in Eq. (2.4), we concentrate our discussion on the properties of the price functions at $u = 0$. 

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2.2 Analytic properties of $V_n(S, t, 0)$

Let $t_0$ denote the grant date of the reload option, then $V_n(S, t, 0)$ is defined only within the time period $[t_0 + \delta, T]$. Assume that at the current time, the time vesting requirement has been satisfied so that $u = 0$. When the time to expiry is less than $\delta$, the holder can exercise at most once, though the option may entitle him to have $n$ reload rights outstanding, where $n > 1$. Strictly speaking, only $V_1(S, t, 0)$ is defined for $t \in [T - t, T]$. For notational convenience, we set $V_n(S, t, 0) = V_1(S, t, 0)$ for $t \in [T - t, T], n \geq 1$. In general, an employee option with $k$ reload rights outstanding can exercise all $k$ reloads only when the time to expiry is not less than $(k - 1)\delta$. As a general rule, we set

$$V_n(S, t, 0) = V_k(S, t, 0), \text{ for } n \geq k \text{ when } t \in [T - k\delta, T].$$

(2.7)

The results in Eq. (2.7) can be rewritten as follows:

$$V_n(S, t, 0) = V_1(S, t, 0) \text{ for } t \in [T - t, T], n \geq 1,$$

(2.8a)

$$V_n(S, t, 0) = V_2(S, t, 0) \text{ for } t \in [T - 2t, T - t], n \geq 2, \text{ etc.}$$

(2.8b)

In this manner, $V_n(S, t, 0)$ is defined for all values of $t$, $t_0 + \delta \leq t \leq T$ (including $n$ being infinite). Since the reload option does not generate cash flows across the critical time instants, $t = T - k\delta, k = 1, 2, \ldots$, the price function $V_n(S, t, 0)$ should be continuous across $t = T - k\delta$ so that

$$V_{k+1}(S, t, 0) = V_k(S, t, 0) \text{ at } t = T - k\delta, k = 1, 2, \ldots.$$

(2.8c)

2.3 Linear complementarity formulation

Since the reload option possesses optimal exercise rights, the pricing model for $V_n(S, t, 0)$ can be formulated as a free boundary value problem. In all subsequent discussions, we take the strike price $X$ to be unity in the pricing formulation for $V_n(S, t, u; X)$. The domain of definition consists of the continuation region and stopping region, which are separated by the free boundary of critical stock price. Let $S_n^*$ denote the critical stock price above which it is optimal to exercise the $n$-reload option. In the continuation region where $S < S_n^*$, $V_n(S, t, 0)$ satisfies the Black-Scholes equation. When $S \geq S_n^*$, the
reload option should be optimally exercised and the exercise payoff is given by Eq. (2.3). Let $\mathcal{L}_{r,q}$ denote the differential operator
\begin{equation}
\mathcal{L}_{r,q} = \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r, \quad r > 0 \text{ and } q \geq 0.
\end{equation}
(2.9)
The linear complementarity formulation for $V_n(S, t, 0)$ is given by
\begin{equation}
\begin{aligned}
&- \left( \frac{\partial V_n}{\partial t} + \mathcal{L}_{r,q} V_n \right) \geq 0, \quad V_n \geq S - 1 + F_{S}^{\mathcal{S}}[V_{n-1}(\tilde{S}, t + \delta, 0)]_{s-1} \\
&\left( \frac{\partial V_n}{\partial t} + \mathcal{L}_{r,q} V_n \right) \left( V_n - \left\{ S - 1 + F_{S}^{\mathcal{S}}[V_{n-1}(\tilde{S}, t + \delta, 0)]_{s-1} \right\} \right) = 0,
\end{aligned}
\end{equation}
(2.10a)
with terminal payoff
\begin{equation}
V_n(S, T; 0) = (S - 1)^+.
\end{equation}
(2.10b)
At maturity, there will be no new reload option received so that the terminal payoff resembles that of the usual call payoff. By virtue of the definition of $V_n(S, t, 0)$ as stated in Eq. (2.7), the above linear complementarity formulation remains valid for all $t \in [t_0 + \delta, T)$.

When $t \in (T - \delta, T)$, the new reload option received upon exercise has no exercise right in its remaining life, so it essentially becomes a European call option. Accordingly, we should interpret $F_{S}^{\mathcal{S}}[V_{n-1}(\tilde{S}, t + \delta, 0)]$ as $C_{E}(S, t)$, for $t \in (T - \delta, T)$, where $C_{E}(S, t)$ denotes the price function of a European call option with unit strike.

### 2.4 Analytic procedures to solve for $V_n(S, t_0, \delta)$

Given the grant date $t_0$, length of vesting period $\delta$, maturity date $T$ and number of allowable reloads $n$, we outline the analytic procedures to solve for $V_n(S, t_0, \delta)$. From Eq. (2.6), we have $V_n(S, t_0, \delta) = F_{S}^{\mathcal{S}}[V_{n}(\tilde{S}, t_0 + \delta, 0)]$ so that it suffices to illustrate the procedures to solve for $V_n(S, t_0 + \delta, 0)$. We consider two separate cases, depending on whether the genuine maximum number of reloads allowable for a given time to expiry is less than $n$ or same as $n$.

1. $t_0 \geq T - n\delta$

   In this case, the time to expiry is less than $n\delta$. Let $k$ denote the genuine allowable maximum number of reloads. Note that $k$ is given
by the largest integer less than \( \frac{T - t_0}{\delta} \), thus \( k \) is less than \( n \) when \( t_0 > T - n\delta \). According to Eq. (2.7), for all \( t \in [t_0 + \delta, T) \), we have \( V_k(S, t, 0) = V_k(S, t, 0) \).

To solve for \( V_k(S, t, 0) \), we first solve for \( V_1(S, t, 0) \) over the last time interval \([T - \delta, T]\). This is a free boundary value problem with exercise payoff \( S - 1 + C_E(1, t) \). Next, we solve for \( V_2(S, t, 0) \) over \([T - 2\delta, T - \delta]\). By continuity of the price function, we have \( V_2(S, T - \delta, 0) = V_1(S, T - \delta, 0) \). As a necessary preliminary step, we obtain \( V_2(S, t, \delta) \) over \([T - 2\delta, T - \delta]\) using Eq. (2.6). This is because the exercise payoff associated with the two-reload option is \( S - 1 + V_1(t, 1, \delta) \). We then proceed the calculations backward in time over each successive time intervals of width \( \delta \). Successively, we solve for \( V_3(S, t, 0) \) over \([T - 3\delta, T - 2\delta]\), \( V_4(S, t, 0) \) over \([T - 4\delta, T - 3\delta]\), etc., until we solve for \( V_k(S, t, 0) \) over the last time interval \([t_0 + \delta, T - (k - 1)\delta]\).

2. \( t_0 < T - n\delta \).

At the first glance, we may proceed as above until we have solved for \( V_j(S, t, 0) \) over the time interval \([T - j\delta, T - (j - 1)\delta], j = 1, 2, \ldots, n - 1 \). Over the time interval \([t_0 + \delta, T - (n - 1)\delta]\), we solve for \( V_n(S, t, 0) \) based on the exercise payoff \( S - 1 + V_{n-1}(1, t, \delta) \). We then encounter a slight complication here. In order to obtain \( V_{n-1}(1, t, \delta) \) over \([t_0 + \delta, T - (n - 1)\delta]\), it is necessary to have the solution for \( V_{n-1}(1, t, 0) \) over \([t_0 + 2\delta, T - (n - 2)\delta]\) so that one has to solve for \( V_{n-1}(S, t, 0) \) over a wider time interval, as compared to the \( \delta \)-width time interval \([T - (n - 1)\delta, T - (n - 2)\delta]\). Proceeding backward to \( V_{n-2}(S, t, 0), \ldots, V_1(S, t, 0) \), we deduce that one has to solve as preliminary steps for \( V_j(S, t, 0), j = 1, 2, \ldots, n - 1 \), over the corresponding wider time interval \([t_0 + (n - j + 1)\delta, T - (j - 1)\delta]\).

We consider two examples to illustrate the above solution procedures:

**Example one: Infinite-reload options**

When the reload option has unlimited reload rights, then only case (1) can occur. We have \( V_\infty(S, t, 0) = V_j(S, t, 0) \) over \([T - j\delta, T - (j - 1)\delta], j = 1, 2, \ldots, \), and \( V_j(S, t, 0), j = 1, 2, \ldots \) are solved successively backward in time over constant time intervals of width \( \delta \) (see Figure 1).
\[ V_\infty (S, t, 0) = V_2 (S, t, 0) = V_1 (S, t, 0) \]
\[ \text{for } T-3\delta \leq t \leq T-2\delta \]
\[ \text{for } T-2\delta \leq t \leq T-\delta \]
\[ \text{for } T-\delta \leq t \leq T \]

**Figure 1** Analytic behaviors of the price function \( V_\infty (S, t, 0) \) of the infinite-reload option over successive time intervals.

**Example two: Two-reload options**

For valuation of a two-reload option granted on date \( t_0 \), we consider the following scenarios:

(a) \( T-\delta \leq t_0 < T \)

The reload right can never be activated so that \( V_2 (S, t_0, \delta) = V_0 (S, t_0, \delta) = C_E (S, t_0) \).

(b) \( T-2\delta \leq t_0 < T-\delta \)

We first solve for \( V_1 (S, t_0 + \delta, 0) \), where \( t_0 + \delta \in [T-\delta, T) \), then we use Eq. (2.6) to obtain

\[ V_2 (S, t_0, \delta) = V_1 (S, t_0, \delta) = F^S_\delta [V_1 (S_t, t_0 + \delta, 0)]. \quad (2.11) \]

(c) \( T-3\delta \leq t_0 < T-2\delta \)

First, we have to solve for \( V_2 (S, t_0 + \delta, 0) \), where \( t_0 + \delta \in [T-2\delta, T-\delta) \).

We obtain \( V_2 (S, t_0 + \delta, 0) \) by solving the optimally stopping problem as governed by Eq. (2.10a,b) with exercise payoff \( S - 1 + V_1 (1, t, \delta) \), \( t \in [T-2\delta, T-\delta) \). Again, \( V_1 (S, t, \delta) \) can be obtained from \( V_1 (S, t + \delta, 0) \) for \( t + \delta \in [T-\delta, T) \) using Eq. (2.6).
(d) \( t_0 < T - 3\delta \)

In this case, we solve for \( V_2(S, t_0 + \delta, 0) \) over the wider time interval \([t_0 + \delta, T - \delta]\). In turns, we first obtain \( V_1(S, t, 0) \) over \([t_0 + 2\delta, T]\) as a preliminary step.

The solution procedures for the valuation of \( V_2(S, t_0, \delta) \), where \( t_0 < T - 3\delta \), is summarized in Figure 2.

\[
\begin{align*}
\text{solve } V_2(S, t, \delta) & \quad \text{over } [t_0 + \delta, T - \delta] \\
\text{solve } V_1(S, t, 0) & \quad \text{over } [t_0 + 2\delta, T]
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
V_2(S, t, \delta) & V_2(S, t, \delta) & V_1(S, t, \delta) \\
= F_\delta [V_2(S, t + \delta, 0)] & = V_1(S, t, \delta) & = C_\delta (S, t) \\
\text{for } t \leq T - 2\delta & \text{for } T - \delta \leq t \leq T & \text{for } T - 2\delta \leq t \leq T - \delta \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
T - 2\delta & T - \delta & T \\
\end{array}
\]

--- calendar time

**Figure 2** Summary of the solution procedures for pricing the two-reload option at the grant date \( t_0 \), where \( t_0 < T - 3\delta \).

### 2.5 Construction of the numerical schemes

Once the continuous pricing model has been well formulated, it becomes quite straightforward to construct the numerical scheme for valuation of the price function of the reload options. We use the standard trinomial tree to simulate the stochastic movement of the stock price. The optimal stopping problem can be solved by the dynamic programming procedure of comparing the continuation value and the exercise payoff at each trinomial node. Let \( m \) denote the number of time steps from option’s maturity, \( j \) be the number
of net upward jumps in stock price in the trinomial random walk. Let $S_{j,m}$
denote the stock price at node $(j,m)$ in the trinomial tree and $\Delta t$ denote
the time step. Let $V_{j,m}^n$ and $\nabla_{j,m}^n$ denote the numerical approximation to
$V_n(S_{j,m}, T - m\Delta t, 0)$ and $V_n(S_{j,m}, T - m\Delta t, \delta)$, respectively, at node $(j,m)$.
The dynamic programming procedure in the trinomial calculations is given by

$$V_{j,m+1}^n = \max(e^{-r\Delta t}(p_u V_{j+1,m}^n + p_m V_{j,m}^n + p_d V_{j-1,m}^n), \nabla_{j,o}^{n-1} + S_{j,m} - 1),$$

(2.12)

where $p_u, p_m$ and $p_d$ denote the probability parameters in the trinomial
scheme, and $j_0$ corresponds to the node index where the stock price equals
the strike price (taken to be unity).

Let the width of the time vesting period $\delta$ corresponds to $d$ time steps,
that is $d \Delta t = \delta$. The numerical valuation of $\nabla_{j,m}^n$ from known solution of
$V_{j,m-d}^n$ amounts to the numerical valuation of the expectation integral defined
in Eq. (2.6). To obtain $\nabla_{j,m}^n$, we perform the usual trinomial calculations
backward in time over $d$ time steps from the $(m-d)^{th}$ time level to the $m^{th}$
time level. These trinomial calculations are performed with the exclusion of the
dynamic programming procedure, and we use the available numerical solution $V_{j,m-d}^n$ as known terminal values in the backward induction calculations.

Compared to the numerical algorithm proposed by Dybvig and Loewenstein
(2003), our algorithm does not have to append an extra index (or di-

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**Numerical valuation of infinite-reload options**

The numerical valuation of the market value of the infinite-reload option $V_{j,m}^\infty$
can be effectively organized as follow.

1. We separate the computational domain into subintervals of $d$ time
steps.

2. In the first $d$ steps, $0 < m \leq d$, we compute $V_{j,m}^1$, using the dynamic
programming procedure as shown in Eq. (2.12). Note that $V_{j,m}^0$ is given
by the value of the European call option counterpart.
3. For the next \(d\) steps, \(d < m \leq 2d\), we first compute \(V^1_{j,m}\) by performing usual backward trinomial calculations (without the dynamic programming procedure) with known terminal values \(V^1_{j,m-d}\), then subsequently compute \(V^2_{j,m}\) using Eq. (2.12).

4. In general, for the \(k\)th subinterval where \((k-1)d < m \leq kd\), we compute \(V^{k-1}_{j,m}\) and \(V^k_{j,m}\) in sequential order.

Note that \(V^\infty_{j,m} = V^k_{j,m}\) when \(m\) satisfies \((k-1)d < m \leq kd\); and by continuity of option value, we have \(V^{k+1}_{j,kd} = V^k_{j,ka}, k = 1, 2, \cdots\).

3 NUMERICAL RESULTS

First, we would like to compare the numerical accuracy of Dybvig-Loewenstein’s algorithm with our algorithm in the numerical valuation of an infinite-reload option. The parameter values used in the pricing model of the infinite-reload option are: \(r = 0.05, q = 0, X = S = 1\) and \(\tau = 10\). Dybvig and Loewenstein also computed the upper bound and lower bound to the option value. The upper bound is given by the value of the infinite-reload option that allows continuous exercise. The lower bound is the value of the infinite-reload option with pre-determined exercise instances, where the interval between successive exercise instances equals the length of the vesting period. Table 1 lists the numerical option values obtained using Dybvig-Loewenstein’s algorithm and our algorithm with varying number of trinomial time steps \(n\), volatility value \(\sigma\) and length of vesting period \(\delta\). The results obtained from our algorithm invariably demonstrate a faster rate of convergence. Numerical results from both algorithms agree with each other within 1% accuracy.

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<th>upper bound</th>
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</tbody>
</table>

Table 1 Comparison of the numerical results obtained from the trinomial calculations using Dybvig-Loewenstein’s algorithm and our algorithm for valuation of an infinite-reload option. The parameter values of the reload option are: \(r = 0.05, q = 0, X = S = 1\) and \(\tau = 10\). Here, \(n\) denotes the total number of time steps used in the trinomial calculations, and \(n = \infty\) refers to the extrapolated option value.
Next, we examine the sensitivity of the option value of infinite-reload options with respect to the length of the time vesting period $\delta$. In Figure 3, we plot the price function (evaluated at $S = 1$) of a 10-year infinite-reload option against $\delta$. The other parameter values of the reload option are: $r = 0.05$ and $q = 0$. The plots agree with the financial intuition that the price function of the infinite-reload option is monotonically decreasing with increasing length of the vesting period and increasing with increasing volatility level.

![Figure 3](image_url)

**Figure 3** We plot the price function $V_\infty(1, 0, \delta)$ of the infinite-reload option against the length of the vesting period $\delta$. The parameter values used in the calculations are: $r = 0.05, q = 0, X = 1, \tau = 10$. We observe that $V_\infty(1, 0, \delta)$ is monotonically decreasing with increasing $\delta$ and increasing with increasing volatility $\sigma$. 

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In Figure 4, we plot the price functions, \( V_1(S, \tau, 0) \) of one-reload option (dashed curve), \( V_2(S, \tau, 0) \) of two-reload option (dot-dashed curve), \( V_3(S, \tau, 0) \) of three-reload option (dotted curve) and \( V_\infty(S, \tau, 0) \) of infinite-reload option (solid curve), against time to expiry \( \tau \), \( \tau = T - t \). Other parameter values used in the calculations are: \( r = 0.05, q = 0, \sigma = 0.4, \delta = 0.5, S = 1 \) and \( X = 1 \). According to Eq. (2.7), all price functions are equal for all \( \tau \in [0, \delta] \). Continuing on with increasing value of \( \tau \), we observe

\[
V_\infty = V_2 = V_3 > V_1 \quad \text{for} \quad \tau \in (\delta, 2\delta]
\]
\[
V_\infty = V_3 > V_2 > V_1 \quad \text{for} \quad \tau \in (2\delta, 3\delta]
\]
\[
V_\infty > V_3 > V_2 > V_1 \quad \text{for} \quad \tau > 3\delta.
\]

For the infinite-reload option, the price function is continuous at \( \tau = \delta, \tau = 2\delta \) and \( \tau = 3\delta \) but exhibits jump in time-derivative at these critical time instants. Recall that \( V_\infty \big|_{\tau=\delta^-} = V_1 \big|_{\tau=\delta^-} \) and \( V_\infty \big|_{\tau=\delta^+} = V_2 \big|_{\tau=\delta^+} \), and we have

\[
\left. \frac{\partial V_\infty}{\partial \tau} \right|_{\tau=\delta^-} = \left. \frac{\partial V_1}{\partial \tau} \right|_{\tau=\delta^-} = \left. \frac{\partial V_2}{\partial \tau} \right|_{\tau=\delta^-} < \left. \frac{\partial V_2}{\partial \tau} \right|_{\tau=\delta^+} = \left. \frac{\partial V_\infty}{\partial \tau} \right|_{\tau=\delta^+},
\]

which agrees with the intuition that a higher rate of increase in option value with increasing \( \tau \) for an option with more reload rights outstanding.

We also examine the sensitivity of the price of a multi-reload option to volatility \( \sigma \) and length of vesting period \( \delta \). In Table 2, we list the numerical option values of one-reload, two-reload and three-reload options. Other parameter values used in the calculations are: \( r = 0.05, q = 0, X = S = 1, \tau = 10 \), and we used 1000 time steps in the numerical calculations. The numerical results reveal that the option price function is an increasing function of \( \sigma \) and a decreasing function of \( \delta \). With a long time to maturity of 10 years, the option values are not quite sensitive to \( \delta \). In particular, the price of one-reload option is independent of \( \delta \) since the time vesting requirement is rendered redundant for options with only one reload right.
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\delta$</th>
<th>one-reload option</th>
<th>two-reload option</th>
<th>three-reload option</th>
</tr>
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<tbody>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>0.5714</td>
<td>0.5945</td>
<td>0.6082</td>
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<tr>
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<td>0.6797</td>
<td>0.6940</td>
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<tr>
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<td>1.0</td>
<td>0.6547</td>
<td>0.6791</td>
<td>0.6923</td>
</tr>
</tbody>
</table>

Table 2: The option values of one-reload, two-reload and three-reload options with varying values of volatility $\sigma$ and length of time vesting period $\delta$. The parameter values used in the calculations are: $r = 0.05, q = 0, X = S = 1$ and $\tau = 0$.

Exercise policies of infinite-reload options

When the employee stock option has unlimited number of reload rights, it behaves like a one-reload option in the time interval $[T - \delta, T]$, a two-reload option in $[T - 2\delta, T - \delta]$, etc. In our earlier paper (Dai and Kwok, 2003), we have proved that when $q = 0$ and $r > \frac{\sigma^2}{2}$, the critical stock price $S^*(\tau)$ for finite-reload options (with no time vesting requirement) only exists for $\tau < \tau^*$, where $\tau^*$ is some threshold value. In Figure 5, we show the plot of $S^*(\tau)$ against $\tau$ for an infinite-reload option subject to time vesting requirement. The parameter values used in the calculations are: $r = 0.1, q = 0, \sigma = 0.1, \delta = 0.5$ and $X = 1$. For $\tau \in [0, 0.5]$, according to the analysis in Appendix A in Dai and Kwok’s paper (2003), the corresponding one-reload option is never optimally exercised when $\tau > \tau_1^* = 0.3982$, where $\tau_1^*$ is the unique solution to the algebraic equation:

$$0.1N(-1.05\sqrt{\tau}) = \frac{0.05}{\sqrt{\tau}} n(-1.05\sqrt{\tau}).$$

For $\tau \in [\tau_1^*, 0.3982, 0.5), S^*(\tau)$ is not defined. That is, it is never optimal to exercise the last reload right when $0.3982 \leq \tau < 0.5$. For $\tau \in [0, 2\delta) = [0.5, 1, 0)$, the infinite-reload option subject to time vesting has essentially two reload rights outstanding. Our numerical calculations show that there exists an interval $[\tau_2^*, 2\delta)$ such that $S^*(\tau)$ does not exist for $\tau \in [\tau_2^*, 2\delta)$ (see Figure 5). Similar behavior occurs in every successive $\delta$-interval: $[(k - 1)\delta, k\delta], k = 1, 2, \ldots$, such that there exists some range of value of $\tau$ where $S^*(\tau)$ is not defined.

Other than the scenario considered in the above ($q = 0, r > \frac{\sigma^2}{2}$ and
the critical stock price for the infinite-reload option subject to time
vesting is a continuous function of \( \tau \). The continuity property of the criti-
cal stock price stems from the continuity of the price function \( V_\infty(S, \tau, 0) \) at
\( \tau = k\delta, k = 1, 2, \cdots \), and absence of cash flows generated from holding the
reload option contract at those time instants. Let \( S_n^*(\tau) \) denote the critical stock
price for a \( n \)-reload option. In Figure 6, we plot \( S_n^*(\tau) \), \( n = 1, 2, 3, \infty \),
against \( \tau \). The set of parameters used in the calculations are: \( r = 0.1, q = 
0, \sigma = 0.3, \delta = 0.5 \) and \( X = 1 \). The critical stock price \( S_n^*(\tau) \) is continuous
at \( \tau = \delta, 2\delta, 3\delta \), etc., but \( S_n^*(\tau) \) is discontinuous at these time instants since
\( V_\infty(S, \tau, 0) \) has discontinuous \( \tau \)-derivative at \( \tau = \delta, 2\delta, 3\delta \) [see Eq. (3.2)].
Since the number of reload rights will decrease from two to one as \( \tau \) de-
creases from \( \tau = \delta^+ \) to \( \tau = \delta^- \), the holder is willing to exercise the reload at
a lower critical stock price as the calendar time is approaching the instant
\( \tau = \delta \). Our calculations also reveal the existence of the limiting value of
\( S_n^*(\tau) \) at infinite time to expiry (see Figure 6).

In Figure 7, we plot \( S_n^*(\infty) \) against \( \delta \) for varying values of \( \sigma \). Other
parameter values used in the calculations are the same as those for Figure 6.
With a longer time vesting period \( \delta \), the holder becomes more conservative
to use the reload rights; so \( S_n^*(\infty) \) increases with increasing \( \delta \). Also, like
the behavior of the critical stock price for the usual American call options,
\( S_n^*(\infty) \) increases with increasing volatility level of the stock prices.
Figure 4 The price functions of one-reload option (dashed curve), two-reload option (dot-dashed curve), three-reload option (dotted curve) and infinite-reload option (solid curve) are plotted against time to expiry $\tau$. The parameter values used in the calculations are: $r = 0.05$, $q = 0$, $\sigma = 0.4$, $\delta = 0.5$, $S = 1$ and $X = 1$. 
Figure 5 The critical stock price $S^*(τ)$ for an infinite-reload option is plotted against time to expiry $τ$. The parameter values used in the calculations are: $r = 0.1, q = 0, \sigma = 0.1, δ = 0.5$ and $X = 1$. There exist some ranges of value of $τ$ where $S^*(τ)$ is not defined, signifying that it is never optimal to exercise the reload feature at these values of $τ$. 

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Figure 6 The critical stock price $S^*_\infty(\tau)$ for an infinite-reload option (labelled “$n = \infty$”) is plotted against time to expiry $\tau$. The plots of $S^*_n(\tau)$, $n = 1, 2, 3$, for an one-reload option (labelled “$n = 1$”), two-reload option (labelled “$n = 2$”) and three-reload option (labelled “$n = 3$”) against $\tau$ are also included for comparison. The parameter values used in the calculations are: $r = 0.1, q = 0, \sigma = 0.3, \delta = 0.5$ and $X = 1$. The curves $S^*_1(\tau), S^*_2(\tau), S^*_3(\tau)$ and $S^*_\infty(\tau)$ overlap with each other over $\tau \in [0, 0.5)$. At $\tau = k\delta, k = 1, 2, 3, \ldots$, $S^*_n(\tau)$ is continuous but its $\tau$-derivative $S^*_{n\tau}(\tau)$ is not continuous.
Figure 7: The limiting value of $S^*_\infty(\tau)$ at infinite time to expiry, $S^*_\infty(\infty)$, is plotted against the time vesting period $\delta$ for varying values of volatility $\sigma$. We observe that $S^*_\infty(\infty)$ is monotonically increasing with respect to both $\delta$ and $\sigma$. 

\[ \sigma = 0.4 \]
\[ \sigma = 0.3 \]
4 CONCLUSION

We have analyzed the impact of the time vesting requirement on the pricing behaviors of employee options with reload rights. An option with unlimited reload rights subject to time vesting requirement is essentially an option with finite reload rights, where the genuine allowable maximum number of reloads decreases as the calendar time is approaching maturity. Whenever the time to expiry falls in value by an amount equals to the length of the vesting period, the price function of the infinite-reload option changes to the price function of a new finite-reload option with one reload right less. The price function of an infinite-reload option exhibits a jump in the time-derivative across those time instants corresponding to time to expiry which equals a multiple of the length of the vesting period.

By incorporating the above analytic properties of the price functions, we have developed an efficient algorithm for numerical valuation of the reload options. Since it becomes unnecessary to append an extra index to track the path dependence associated with the vesting time, the computational complexity of our algorithm is the same order as that of reload option without the time vesting requirement.

We also have examined the optimal decision of exercising the reload for infinite-reload options subject to time vesting requirement. Depending on the parameter values in the pricing model, the critical stock price may or may not be defined for all times. At those times where the critical stock price is not defined, the non-existence of the critical stock price signifies that it is never optimal to exercise the reload at any stock price level. For the perpetual infinite-reload options, the critical stock price increases with increasing length of the time vesting period and increasing volatility level of the stock price.

REFERENCES


