Continuum Damage Mechanics Based Modeling of Fiber Reinforced Concrete in Tension

by

Faming Li and Zongjin Li

Department of Civil Engineering

The Hong Kong University of Science and Technology

July 21, 1999

Revised Version

Submitted to *The International Journal of Solids and Structures*, July 1999.

Continuum Damage Mechanics Based Modeling of Fiber Reinforced Concrete in Tension

Faming Li* and Zongjin Li†

Department of Civil Engineering
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Abstract

The tensile properties of concrete can be enhanced substantially by incorporating high strength and small diameter short fibers which leads to fiber reinforced concrete (FRC). For this reason, FRC has been widely used in infrastructures where tensile cracks may occur. However, an analytical model for such a material is still lacking. In this paper, an attempt was made to model the behavior of FRC shows a hardening response in tension, based on the Continuum Damage Mechanics (CDM). In the material, conventional concrete (a cement-sand-coarse-aggregate-water mix) was used as the matrix and short steel fibers were used as the reinforcement. The quasi-brittleness of the matrix and the fiber-matrix interfacial properties were taken into consideration. Results show that the model-predicted stress-strain curves agree well with those obtained experimentally.

1 Introduction

In a conventional FRC, the fiber content is usually within the range of 0.2-2% by volume (Zollo, 1997). At such a lower fiber content, the tensile response of FRC would assume a non-hardening type, which is characterized by the widening of a single crack,

*PhD, Department of Civil Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: celfm@ust.hk

†Author to whom correspondence should be addressed. Assistant Professor, Department of Civil Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: zongjin@ust.hk

assume a non-hardening type, which is characterized by the widening of a single crack, similar to an un-reinforced concrete (Li, 1998). It has been reported that strain hardening and multiple cracking behavior can be achieved when aligned continuous fibers were used in the composite (Aveston et al., 1971; Li et al., 1992). Such a phenomenon was also observed in cementitious composites containing a higher volume fraction (say, 8%) of randomly distributed short fibers (Naaman and Homrich, 1989). Analytical models of this kind of behavior were also developed (Li and Leung, 1992). In these composites, the fibers can, during loading, provide sufficient bridging forces to suppress crack opening and strain localization, and to yield multiple cracks. It should be pointed out, however, that the matrix of the investigated composite was either cement paste, mortar or cementitious slurry. Coarse aggregates were not incorporated.

Recently, it has been reported that FRC containing short fibers can also achieve a strain hardening type response when fiber content is high enough, in which conventional concrete was used as the matrix and short steel fibers were used as the reinforcement (Li et al., 1998). For these FRCs, since the concrete matrix is a quasi-brittle material, which is mechanically different from a cement paste or mortar matrix that can be approximated as a brittle material, the conventional linear elastic fracture mechanics or micromechanics cannot be directly applied to model their behavior (Shah et al., 1995).

In this paper, a nonlinear analytical model based on the concept of CDM is developed to characterize the tensile stress-strain response of FRC. A parallel bar arrangement of the composite is first used to establish the equilibrium equation. A basic governing equation is then derived coupling with the damage law of the matrix and that of the fibers. The damage evolution is then evaluated by considering the damage evolution of the matrix concrete and that of the fiber-matrix interface, respectively. Numerical results obtained from the constitutive equations of this model are compared to those obtained from experimental investigations.

2 The Analytical Model

Since the mechanical response of an FRC is closely related to the nucleation and progression of a multitude of microcracks in the matrix, the Continuum Damage Mechanics (CDM) provides a possibly suitable method for developing an analytical model of the mechanical behavior of an FRC. The concept of CDM has been successfully applied to plain concrete (see, *e.g.*, Karihaloo and Fu, 1990a; Karihaloo and Fu, 1990b; Løland, 1980; Mazars and Pijaudier-Cabot, 1989). Attempts have also been made to apply this concept to the modeling of a fiber reinforced concrete (see, *e.g.*, Borderie et al., 1992; Fanella and Krajcinovic, 1985; Stang, Tonnesen and Byskov, 1990). A one-dimensional analytical model for the direct tensile response of an FRC based on the concept of CDM will be described in this paper. It is assumed that before the matrix damage is initiated, the material behaves as a two-phase composite material. As deformation increases, the matrix and the fiber will undergo independently different damage processes: the matrix concrete responds as a plain concrete, whereas the damage of the fiber evolves with the fiber debonding, slip and pull-out of the matrix. Thus, the damage in the cross-section is measured by the loss of the cross-sectional "area" of the matrix and the bond variation of the fiber-matrix interface.

The parallel bar model has been previously employed in the study of plastic and brittle behavior of materials (Krajcinovic and Silva, 1982). It has also been used to analyze the mechanical behavior of fiber reinforced concrete by Fanella and Krajcinovic (1985). In that model, it is assumed that once the tensile stress in the matrix is higher than its tensile strength, the tensile load of the composite bar is transmitted to the fibers alone, which implies that the matrix is a purely brittle material. This is not the case for FRC using conventional concrete as the matrix. Therefore, in applying the

model to the analysis of FRCs, it has to be modified because of the quasi-brittleness of the concrete matrix.

Consider the parallel bar model shown in Fig. 1. In the figure, L is the length of the specimen for a single cracking system, or the crack spacing for a multiple cracking system. Every bar is a completely elastic composite until the matrix damage is initiated. After the initiation of the matrix damage, the stiffness of such a bar decreases progressively, due to the stiffness reduction of the matrix as well as the fiber debonding and slip.

Thus, the force f_i in the i th bar is

$$f_i = \frac{k_c z}{N} \quad \text{for } 0 \leq k_c z \leq f_m \text{ and } z \geq 0 \quad k_c \quad (1)$$

$$f_i = \frac{k_d z}{N} \quad \text{for } k_c z \geq f_m \text{ and } z \geq 0 \quad k_d \quad (2)$$

in which N = the number of composite bars; k_c/N = a constant, the stiffness of each of N composite bars in which no damage occurred; k_d/N = the stiffness of the bar in which the matrix has damaged, which is a random variable; f_m = the force at which the matrix damage is initiated, and z = the extension of the total specimen due to the active force F .

Consider the force and deformation in each of the composite bars in which the matrix has damaged. A parallel model, as has been commonly used in the composite law, is assumed for the damaged matrix and the fibers, as shown in the enlarged part of Fig. 1. However, the fiber-matrix interface is no longer "perfectly bonded", because of the fiber debonding and sliding. Suppose the elongation due to the fiber (debonding and sliding) is $\alpha_1 z$ and that due to the matrix is $\alpha_2 z$, then we have

$$f_i = \frac{k'_f \alpha_1 z}{MN} + \frac{k_m \alpha_2 z}{MS} \quad (3)$$

where k'_f = an equivalent stiffness of the fiber that elongates $\alpha_1 z$, including the elastic elongation and the slip from the matrix; k_m = the stiffness of the damaged matrix, which is a random variable depending upon the damage accumulation. M = the total number of fibers in each of the N composite bars; S = total number of matrix bars in each of the N composite bars; α_1 and α_2 are coefficients reflecting the deformation components of the fiber and the matrix, respectively, which will be discussed later. Note that $\alpha_1 \neq \alpha_2$ in this model.

The equilibrium of forces in the vertical direction yields

$$F = \sum_{n+1}^N \frac{k_c z}{N} + \sum_1^n \left[\sum_{m+1}^M \frac{k'_f \alpha_1 z}{NM} + \sum_{s+1}^S \frac{k_m \alpha_2 z}{SM} \right] \quad (4)$$

in which n = the number of composite bars in which the matrix has damaged; m = the number of fibers that failed due to debonding and pull-out in the damaged matrix; and s = the number of damaged matrix bars. The first term on the right hand side of Eq. (4) represents the contribution of the integral (undamaged) part of the composite cross-section while the second term reflects the tensile force carried by the fibers spanning the damaged matrix and the quasi-brittleness of the matrix (for a brittle matrix, the last term can be assumed zero, as has been done by Fanella and Krajcinovic 1985).

Eq. (4) can be further written as

$$F = k_c z (1 - \omega_c) + k'_f z \alpha_1 \omega_c (1 - \omega_f) + k_{m0} z \alpha_2 \omega_c (1 - \omega_c)^2 \quad (5)$$

in which

$$\omega_c = \frac{n}{N} \approx \frac{s}{S}, \quad \omega_f = \frac{m}{M} \quad (6)$$

k_{m0} = the stiffness of the matrix material before damage, and

$$k_m = k_{m0} (1 - \omega_c) \quad (7)$$

ω_c and ω_f are the logical choices for the measure of damage accumulated in the cross-section of the FRC specimen, which denote the damage of the composite due to the matrix damage and the fiber damage (debonding and pullout), respectively. In Eq. (6), it is assumed that the fibers and aggregates are uniformly distributed in the composites.

The continuum form of Eq. (5) can thus be readily recovered by setting $F = \sigma A_c$, $k_c = E_c A_c / L$, $k'_f = \beta_f k_c$, $k_{m0} = \beta_m k_c$, and $z = \varepsilon L$ in which σ = nominal stress. The term ε is the nominal strain, and β_f and β_m are the stiffness ratios. Thus

$$\sigma = E_c \left[(1 - \omega_c) + \beta_f \alpha_1 \omega_c (1 - \omega_f) + \beta_m \alpha_2 \omega_c (1 - \omega_c)^2 \right] \varepsilon \quad (8)$$

A clear implication can be seen from Eq. (8) that when $\omega_c = 0$, or no damage is initiated, $\sigma = E_c \varepsilon$, *i.e.*, the composite behaves elastically; when $\omega_c = 1$, or matrix is completely failed, $\sigma = E_c \beta_f \alpha_1 (1 - \omega_f) \varepsilon$, the contribution of the matrix disappeared; and when $\omega_c = \omega_f = 1$, or both the matrix is completely failed and the fibers are completely pulled out, $\sigma = 0$, *i.e.*, the composite cannot bear any load. Therefore, a rational stress-strain relationship would be obtained by this model.

3 Evolution of Damage

3.1 Evolution of ω_c

Damage ω_c can be characterized by the cumulative degradation in the load carrying capacity of the matrix. As mentioned earlier, $\omega_c = 0$ denotes no damage in the matrix, while $\omega_c = 1$ represents that the matrix failed completely. Therefore, ω_c can be approximately represented by considering the damage evolution of the matrix concrete.

There are a lot of CDM-based models for plain concrete. To make things not too complicated, a simple local approach proposed by Løland (1980) will be used in this paper. However, a minor modification is made to accommodate the present study.

Assume that there is no damage below a strain level ε_0 . The damage evolution is evaluated as

$$\omega_c = \begin{cases} 0 & \text{for } \varepsilon < \varepsilon_0 \\ B_1(\varepsilon - \varepsilon_0)^\xi & \text{for } \varepsilon_0 \leq \varepsilon \leq \varepsilon_t \\ \omega_{ct} + B_2(\alpha_2\varepsilon - \varepsilon_t) & \text{for } \varepsilon_t < \varepsilon \leq \varepsilon_u/\alpha_{2m} \\ 1 & \text{for } \varepsilon_m > \varepsilon_u/\alpha_{2m} \end{cases} \quad (9)$$

in which ε is the composite strain, α_{2m} is a α_2 when a matrix strain ε_m reaches ε_u (fictive ultimate strain), where ε_m is the strain of the matrix concrete. ε_0 is the strain at which damage is initiated, ε_t is the strain corresponding to the tensile strength, f_{mt} , of the concrete, and ω_{ct} is the damage of concrete corresponding to a strain of ε_t . ω_{ct} and the constants B_1 , B_2 , and ξ can be determined by

$$\omega_{ct} = 1 - \frac{f_{mt}}{E_{m0}\varepsilon_t} \quad (10)$$

$$\xi = \frac{\zeta}{(1-\zeta)t}, \quad B_1 = \frac{1}{1+\xi t}(\varepsilon_t - \varepsilon_0)^{-\xi}, \quad B_2 = \frac{\zeta t}{\varepsilon_t} \quad (11)$$

in which

$$\zeta = \frac{f_{mt}}{E_{m0}\varepsilon_t}, \quad t = \frac{\varepsilon_t}{\varepsilon_u - \varepsilon_t} \quad (12)$$

The damage evolution and the stress-strain curve are roughly shown in Fig. 2. The denotations of ε_0 , ε_t , ε_u , and ω_{ct} can be found in that figure. In the above equations for ω_c , there are five essential parameters, namely, E_{m0} , f_{mt} , ε_0 , ε_t , and ε_u . Except for ε_u , which is a fictive ultimate strain, all other parameters can be determined from the experimentally obtained stress-strain curves in a direct tensile test. Løland (1980) argued that the energy ratio between the specific energy consumed in microcrack formation within the whole strained body (length L), and that consumed

in further cracking of the fracture zone (with extension of D_{max}), remains constant, and independent of the water-cement, and aggregate-cement ratios used. This assumption leads to

$$\varepsilon_u = \frac{2}{R} \frac{D_{max}}{L} \left(\varepsilon_t - \frac{f_{mt}}{E_{m0}} \right) + \varepsilon_t \quad (13)$$

in which R is the aforementioned energy ratio, which can be determined from experiments.

3.2 Evolution of ω_f

Damage ω_f characterizes the damage of the fiber in a composite bar of damaged matrix, namely, the fraction of fibers that are debonded and pulled out. If no fiber is debonded, the fiber and the matrix underwent a same deformation. If all fibers are debonded and pulled out, the composite bar fails completely.

Damage ω_f can therefore be related to the fiber debond length a and fiber embedment length $\bar{\ell}_e$ as

$$\omega_f = \frac{a}{\bar{\ell}_e} \quad (14)$$

where $\bar{\ell}_e$ is the mean fiber embedment length. From a statistical consideration, $\bar{\ell}_e = \ell/4$ for a random fiber distribution, where ℓ is the length of the fiber.

Eq. (14) implies that when the fiber debond length $a = 0$, no damage occurs, while for $a = \bar{\ell}_e$, the fiber is completely debonded and pulled out ($\omega_f = 1$). Therefore, damage ω_f can characterize the failure of the fiber due to the interface debonding and fiber pullout.

To evaluate the evolution of damage ω_f , it is necessary to examine the fiber matrix interfacial behavior as well as the cumulative distribution of the fiber embedment length. A fiber-matrix interfacial model has been examined in detail by Stang

et al (1990) in which the effect of the weak transition zone existing between ordinary Portland cement paste and the fiber was taken into consideration. In the model, it is assumed that a pre-debonded interface (interfacial crack) exists with a length a (starting at where the fiber and matrix meet) in the fiber-matrix interface. The fiber with a constant cross-sectional area of A (with a diameter of d_f) and Young's modulus of E_f is embedded in the matrix with a mean embedment length of $\bar{\ell}_e$. The fiber axial displacement is denoted by U and is assumed to be constant over the fiber cross section. The bonded zone is assumed to be elastic with a stiffness k , while the debonded zone is represented by a constant shear force q_f . The Poisson's ratio is neglected for both fiber and the boundary layer (*i.e.*, the shear lag). According to this model, the displacement at the fiber free end U^* can be written as

$$U^* = \frac{q_f D}{E_f A \psi^2} + \frac{q_f \bar{\ell}_e^2}{2 E_f A} \left(\frac{a}{\bar{\ell}_e} \right)^2 + \frac{q_f \bar{\ell}_e D}{E_f A \psi} \left(\frac{a}{\bar{\ell}_e} \right) \tanh \left[\psi \bar{\ell}_e \left(1 - \frac{a}{\bar{\ell}_e} \right) \right] \quad (15)$$

in which D and ψ are defined as

$$D = \frac{1}{2} + \sqrt{\left(\frac{1}{2} \right)^2 + \frac{2\pi d_f k \Gamma}{q_f^2}}, \quad \psi = \sqrt{\frac{k}{E_f A}} \quad (16)$$

and Γ is the interface critical energy release rate.

At the applied load F , the damaged composite bar will elongate z . Before the matrix cracks, it can be assumed that there is no fiber slip in the matrix (although damage may have been initiated). Then the deformation of the matrix is approximately equal to the elongation of the composite, z , *i.e.*, $\alpha_2 = 1$, and the contribution of fibers to the composite elongation can be neglected, or $\alpha_1 \approx 0$. After the matrix cracks, α_1 will normally be not equal to α_2 . The deformation of the fiber and the matrix can then be written as

$$\begin{aligned} \alpha_1 z &= 2 \cdot U^* \\ \alpha_2 z &= \epsilon_m \cdot L \end{aligned} \quad (17)$$

Combining Eqs. (17), (15) and (14), it can be obtained that

$$\frac{L}{2} \cdot \alpha_1 \varepsilon = \frac{q_f D}{E_f A \psi^2} + \frac{q_f \bar{\ell}_e^2}{2 E_f A} \omega_f^2 + \frac{q_f \bar{\ell}_e D}{E_f A \psi} \omega_f \tanh \left[\psi \bar{\ell}_e (1 - \omega_f) \right] \quad (18)$$

provided that

$$\varepsilon > \varepsilon_0^f = \frac{2 q_f D}{\alpha_{10} L E_f A \psi^2} \quad (19)$$

which suggests that no damage related to fibers occurs before debonding is initiated. In Eq. (19), α_{10} is the α_1 corresponding to the initial debonding strain ε_0^f . ω_f can be obtained numerically by solving Eq. (18).

Once ω_c and ω_f are determined, the continuum model can then be established through Eq. (8). Parameters related to the model will be discussed in the following section.

4 Determination of the Parameters

Several parameters have to be determined before the application of the derived model. These include: (1) stiffness ratios β_f and β_m ; (2) composite elastic modulus E_c ; (3) length L ; (4) the fiber-matrix interface parameters: ψ , q_f , and Γ ; (5) the parameters related to matrix concrete: ε_0 , ε_t , ε_u , f_{mt} , and E_{m0} ; (6) damage related to fibers ω_f , and (7) coefficients α_1 and α_2 ;

4.1 Stiffness ratios β_f and β_m

An exact estimation of β_f will be very difficult, because the equivalent stiffness of the fiber varied during the process of fiber debonding and slipping. However, an average stiffness can be estimated by the following analysis.

Consider a fiber with an average embedment length of $\ell/4$ that bridges a transverse crack. One half of this embedment length is debonded, in a statistical sense, and the other half is perfectly bonded through elastic adhesional bond. The stress transfer between the fiber and the matrix will cause an average force build-up in the fiber, which can be derived as (set $a = \ell/8$), according to the interface model (Stang, Li and Shah, 1990):

$$\bar{P}_f = \frac{4q_f D}{\psi^2 \ell \cosh \frac{\psi \ell}{8}} \left[\cosh \left(\frac{\psi \ell}{8} \right) - 1 \right] + \frac{3q_f \ell}{32} \quad (20)$$

The average deformation in the fiber, by taking into consideration of the elastic deformation of the fiber only, can be derived as

$$\bar{U}_f = \frac{4q_f D}{E_f A \psi^3 \ell} \left[1 + \frac{3(\psi \ell)^2}{128} \right] \tanh \left(\frac{\psi \ell}{8} \right) \quad (21)$$

The average equivalent stiffness for a single fiber can be therefore estimated as

$$\frac{k'_f}{MN} = \frac{\bar{P}_f}{\bar{U}_f} = \frac{E_f A \psi}{128 + 3(\psi \ell)^2} \left[128 \tanh \left(\frac{\psi \ell}{16} \right) + \left(\frac{3}{D} \right) (\psi \ell)^2 \coth \left(\frac{\psi \ell}{8} \right) \right] \quad (22)$$

The total number of fibers MN in a composite cross section will be (for 2-D random distribution) (Hannant, 1978)

$$MN = \frac{2V_f A_c}{\pi A} \quad (23)$$

Therefore,

$$\begin{aligned} \beta_f &= \frac{\ell}{2\ell_c} \cdot \frac{k'_f}{k_c} = \frac{\ell}{2\ell_c} \cdot \frac{k'_f}{E_c A_c / L} \\ &= \frac{\ell E_f V_f \psi L}{\pi E_c \ell_c (128 + 3\psi^2 \ell^2)} \left[128 \tanh \left(\frac{\psi \ell}{16} \right) + \left(\frac{3}{D} \right) \psi^2 \ell^2 \coth \left(\frac{\psi \ell}{8} \right) \right] \end{aligned} \quad (24)$$

in which a length efficiency factor $\ell/(2\ell_c)$ is incorporated.

The stiffness ratio β_m can be estimated from its definition as

$$\beta_m = \frac{k_{m0}}{k_c} = \frac{A_m E_{m0}/L}{A_c E_c/L} = \frac{E_{m0}}{E_c} (1 - V_f) \quad (25)$$

4.2 Composite Elastic Modulus E_c

For a composite containing short aligned fibers, a “Rule of Averages” is used (see, Hull and Clyne, 1996) to derive the composite modulus, which can be expressed as

$$E_c = V_f E_f \left[1 - \frac{\tanh(n_r \ell/d_f)}{n_r \ell/d_f} \right] + (1 - V_f) E_{m0} \quad (26)$$

However, for composite containing randomly distributed short fibers, Eq. (26) has to be modified. Taking account of the fiber orientation effect, η , Eq. (26) can be modified as

$$E_c = \gamma V_f E_f + (1 - V_f) E_{m0} \quad (27)$$

where

$$\gamma = \eta \left[1 - \frac{\tanh(n_r \ell/d_f)}{n_r \ell/d_f} \right] \quad (28)$$

in which η is the fiber orientation efficiency factor defined as (Bentur and Mindess, 1990):

$$\eta = \begin{cases} \frac{1}{6} & \text{for fully random fibers in 3-D} \\ \frac{1}{3} & \text{for random fibers in 2-D} \\ \frac{1}{2} & \text{for planar mat with aligned fibers in } XY \text{ direction} \end{cases} \quad (29)$$

and n_r is a dimensionless constant given by

$$n_r = \left[\frac{2E_{m0}}{E_f (1 + \nu_m) \ln(1/V_f)} \right]^{1/2} \quad (30)$$

where ν_m is the Poisson's ratio of the matrix.

4.3 Length L

The length L is the crack spacing in a multiple cracking system. The crack spacing is difficult to measure experimentally. Therefore, an analytical result is used instead in this study.

The ACK model (Aveston et al., 1971) for continuous aligned fiber composite assuming frictional stress transfer gives out a minimum crack spacing of

$$\ell_s = \left(\frac{V_m}{V_f} \right) \frac{f_{mt} d_f}{4\tau_{fu}} \quad (31)$$

where τ_{fu} is the frictional bond strength of the fiber-matrix interface, and f_{mt} is the tensile strength of the concrete matrix. After crack saturation (no further crack will be formed), a final crack spacing of between ℓ_s and $2\ell_s$ is expected. An average crack spacing of $1.337\ell_s$ is obtained by Kimber and Keer (1982).

Taking into consideration of the fiber length effect and the orientation effect (Bentur and Mindess, 1990), this average crack spacing becomes

$$\begin{aligned} L &= 1.337 \left(\frac{2\ell_c}{\ell} \right) \left(\frac{\pi}{2} \right) \frac{V_m f_{mt} d_f}{V_f 4\tau_{fu}} \\ &= 1.050 \frac{\ell_c}{\ell} \frac{V_m f_{mt} d_f}{V_f \tau_{fu}} \end{aligned} \quad (32)$$

where ℓ_c is the critical fiber length, which is given by

$$\ell_c = \frac{\sigma_{fu} \cdot d_f}{2\tau_{fu}} \quad (33)$$

in which σ_{fu} is the ultimate tensile strength of the fiber, which can be found in Table 1 for the fibers used in this study.

4.4 Interface parameters ψ , q_f , and Γ

Numerous efforts have been made with respect to the fiber-matrix interfacial properties, both experimentally and analytically. The interface parameters ψ , q_f , and Γ are therefore best determined based on experimental results. Unfortunately, there is no direct measurement of these parameters from the literature, and very few experimental investigations into hooked fibers (Dramix fibers in this study) were made. Therefore, these parameters were determined rationally based on previous studies (e.g., Bentur and Mindess, 1990; Gray, 1984; Li et al., 1991; Pompo et al., 1996). According to Li et al. (1991), parameter ψ is a material property independent of fiber embedment length. Therefore, for Dramix steel fibers used in this study, the following values were used: $\psi = 90 \text{ m}^{-1}$, $\tau_{fu} = 3.80 \text{ MPa}$, $q_f = \tau_{fu} \cdot \pi d_f = 5969 \text{ N/m}$, $\Gamma = 17.2 \text{ N/m}$.

4.5 Matrix parameters ε_0 , ε_t , ε_u , f_{mt} and E_{m0}

The properties of the matrix concrete have been studied extensively. These parameters are determined from the experimental results of these studies. The results shown in Table 2 are obtained based on direct tensile tests on ten specimens which had mix proportions of C:W:S:A=1:0.45:2:0.65.

Note that ε_0 is measured as the strain corresponding to stress of $40\%f_{mt}$ from the stress-strain curve. This is based on observations from the acoustic emission measurement that the microcracking is initiated at a stress of about $40\%f_{mt}$, as described by Li (1998) and Li and Li (1999). Strain is measured as the average displacement over the gage length of the LVDT, where the average displacement is based on those detected by the two opposite LVDT transducers among which one of them is used as feedback control. In addition, a Poisson's ratio $\nu_m = 0.17$ is used for the matrix concrete, as shown by Gray (1984).

4.6 Damage Related to Fibers ω_f

The measure of damage related to fibers, ω_f , is related to the composite strain by Eq. (18), from which ω_f can be solved numerically when $\varepsilon > \varepsilon_0^f$. An extreme value of ε exists for Eq. (18), representing a moment at which successive fiber debonding will be terminated. The corresponding ω_{fc} , a critical value representing the termination of successive fiber debonding beyond which adhesional bond is completely deteriorated, can be obtained by taking a first order derivative of Eq. (18) to ω_f , and let it be zero as

$$\begin{aligned} \omega_{fc} \cdot \left\{ \tanh \left[\psi \bar{\ell}_e (1 - \omega_{fc}) \right] \right\}^2 \\ + \frac{1}{\psi \bar{\ell}_e} \tanh \left[\psi \bar{\ell}_e (1 - \omega_{fc}) \right] - \left(1 - \frac{1}{D} \right) \cdot \omega_{fc} = 0 \end{aligned} \quad (34)$$

which can also be obtained numerically. The existence of a critical damage, ω_{fc} , can be attributed to the ideal interface property assumption. When the bonded length of the fiber is reduced to a critical length, the adhesional bond will deteriorate completely such that only frictional bond exists. Therefore, ω_{fc} can be used as an indication of the maximum load the composite can carry.

4.7 Coefficients α_1 , α_2 , α_{10} and Initial Debonding Strain ε_0^f

As mentioned previously, coefficients α_1 and α_2 represent the composite deformation contribution from the fiber and from the matrix, respectively. It is not quite understood how much fractions of deformation are distributed for the two phases after the matrix cracks. However, the following boundary conditions have to be satisfied:

$$\begin{aligned} (1) \quad & \alpha_1 \approx 0, \quad \alpha_2 = 1 \quad \text{for } \varepsilon \leq \varepsilon_t \\ (2) \quad & \alpha_1 \leq 1, \quad \alpha_2 \leq 1 \quad \text{for } \varepsilon > \varepsilon_t \end{aligned} \quad (35)$$

which implies that before the matrix cracks, the fibers have approximately no contribution to the composite deformation. After that, they both contribute to the composite deformation.

Naaman and Homrich (1989) proposed an estimation of the strain capacity (peak point strain of the tensile stress-strain curve), ε_{ca} as:

$$\varepsilon_{ca} = \varepsilon_u + KV_f \frac{\ell}{d_f} \quad (36)$$

in which ε_{ca} is the strain at maximum tensile stress of the composite, ε_u is the ultimate tensile strain of the unreinforced matrix, and K is a constant depending on the fiber type, which has to be determined from experiments. Based on the experimental results from this study, $\varepsilon_u = 406.3 \times 10^{-6}$ for matrix concrete, and

$$K = \begin{cases} 0.00099 & \text{for Dramix I steel fiber} \\ 0.00115 & \text{for Dramix II steel fiber} \end{cases} \quad (37)$$

Eq. (36) implies that the strain capacity of the composite is composed of two parts: the contribution from the matrix ε_u and the contribution from the fibers $KV_f \ell / d_f$. It is therefore evident that

$$\begin{aligned} (3) \quad \alpha_1 &= \alpha_{1m} = \frac{KV_f \ell / d_f}{\varepsilon_{ca}} \\ (4) \quad \alpha_2 &= \alpha_{2m} = \frac{\varepsilon_u}{\varepsilon_{ca}} \end{aligned} \quad \text{when } \varepsilon = \varepsilon_{ca} \quad (38)$$

in which

$$\alpha_1 + \alpha_2 = 1 \quad (39)$$

is implied.

The deformation contribution of the matrix, after the matrix cracks, will largely come from the post-peak opening of the crack (softening regime of the stress-strain

curve). However, as has been examined by Balagura and Shah (1992), the response of the matrix in an FRC varied with the content and type of the reinforcement. Therefore, it is rational to assume, based on the assumption of the fictive ultimate strain, as shown in Fig. 2, that

$$\alpha_2 = g \cdot \frac{\varepsilon_u - \varepsilon_m}{\varepsilon_u - \varepsilon_t} = \frac{\varepsilon_m}{\varepsilon} \quad (40)$$

where g is a constant dependent on the fiber volume fraction V_f and the type of the fiber used in the FRC, which has to be determined from experimental results. In this study, it is obtained that

$$g = \begin{cases} 2.05 - 2.5V_f & \text{for Dramix I steel fiber, } V_f \leq 9\% \\ 3.85 - 45.0V_f & \text{for Dramix II steel fiber, } V_f \leq 5\% \end{cases} \quad (41)$$

Therefore, strain in the matrix ε_m can be estimated by

$$\varepsilon_m = \frac{\varepsilon_u}{1 + \frac{\varepsilon_u - \varepsilon_t}{g\varepsilon}} \quad (42)$$

which implies that when $\varepsilon = \varepsilon_t$, and $g = 1$, then $\alpha_2 = 1$. Whereas when $\varepsilon = \varepsilon_{ca} \gg (\varepsilon_u - \varepsilon_t)$, $\varepsilon_m \approx \varepsilon_u$, and $\alpha_2 \approx \alpha_{2m}$. Thus it meets the boundary conditions (Eqs. (35) and (38)), if g is set at 1 when $\varepsilon \leq \varepsilon_t$.

Eqs. (40) and (42) implies that

$$\alpha_2 = \frac{\varepsilon_u}{\varepsilon + \frac{\varepsilon_u - \varepsilon_t}{g}} \quad (43)$$

Therefore, from Eq. (39), we have

$$\alpha_1 = \frac{g\varepsilon + (1 - g)\varepsilon_u - \varepsilon_t}{g\varepsilon + \varepsilon_u - \varepsilon_t} \quad (44)$$

The coefficient α_{10} and initial debonding strain ε_0^f can then be evaluated as follows. From Eq. (44), we have

$$\alpha_{10} = \frac{g\varepsilon_0^f + (1-g)\varepsilon_u - \varepsilon_t}{g\varepsilon_0^f + \varepsilon_u - \varepsilon_t} \quad (45)$$

Eq. (19) can be rewritten as

$$\alpha_{10}\varepsilon_0^f = \frac{2q_f D}{LE_f A \psi^2} \quad (46)$$

Thus, α_{10} and ε_0^f can then be obtained numerically by solving an equation combining Eqs. (45) and (46).

5 Numerical Results

The constitutive model characterized by Eq. (8) for the prediction of tensile stress-strain curves of FRC containing Dramix steel fibers shows strain hardening responses, was tested against the experimental results obtained in this study. Parameters needed for the model are given to a large extent in the previous sections. Some of them, calculated by the corresponding equations, are given in Table 3. Comparison between the model predicted stress-strain relationship and that obtained from experimental investigations are shown in Figs. 3, 4, and 5, for FRC containing Dramix type I steel fiber at a V_f of 6%, 7%, and 8%, respectively. Those for FRCs containing Dramix type II steel fibers are shown in Figs. 6, 7, and 8, at a fiber volume fraction of 3%, 3.5%, and 4%, respectively. Parameters related to the fiber-matrix interface (ψ , k and Γ) are selected based on results from references. Those related to the matrix are determined experimentally (Li et al., 1998; Li, 1998). Note that only the hardening regime of the composite response are plotted in these figures.

These figures show a good agreement between the experimental results and the model predicted results, except for FRC containing Dramix type II steel fibers at a

$V_f=3\%$ and 3.5% . At a comparatively lower fiber content, the post-cracking behavior of a composite will be dominated by the fiber pull-out from the matrix along the transverse crack. For Dramix type II steel fibers, anchorage effect provided by the hooked ends is not quite understood, thus not incorporated in the model, which may be reflected in the model predicted results.

6 Summary and Conclusions

In this paper, an analytical model for the tensile behavior of an FRC shows a strain hardening response is developed based on the principles of the continuum damage mechanics. Assuming a parallel bar model in which the fibers and the concrete were connected by parallel-series components, an equilibrium equation was established for the FRC. By taking two damage measures related to the matrix and to the fibers respectively, a continuum form of the equation was then established. The evolution of the two measures of damage was then evaluated in detail. The fiber-matrix interface properties are incorporated explicitly in the model.

A good agreement between the model predicted stress-strain curves and the experimentally obtained shows that the analytical model derived from the principles of the continuum damage mechanics is quite successful. The model requires the knowledge of the matrix (five parameters) and of the fiber-matrix interface properties (three parameters), which has to be determined experimentally.

Acknowledgments

This research is sponsored by the Research Grants Council of the Hong Kong Government under the grant HKUST641/95E which is gratefully acknowledged.

References

- Aveston, J., Cooper, G. A. and Kelly, A. 1971. Single and multiple fracture, *The Properties of Fiber Composites*, National Physical Laboratory, IPC Science and Technology Press Ltd., pp. 15–26.
- Balaguru, P. N. and Shah, S. P. 1992. *Fibre Reinforced Cement Composites*, McGraw-Hill Book Co., New York.
- Bentur, A. and Mindess, S. 1990. *Fibre Reinforced Cementitious Composites*, Elsevier Applied Science, London.
- Borderie, C. L., Mazars, J. and Pijaudier-Cabot, G. 1992. Response of plain and reinforced concrete structures under cyclic loadings, *Concrete Design Based on Fracture Mechanics*, ACI Publication SP-134, Edited by Walter Gerstle and Zdeněk P. Bažant, American Concrete Institute, pp. 147–172.
- Fanella, D. and Krajcinovic, D. 1985. Continuum damage mechanics of fiber reinforced concrete, *Journal of Engineering Mechanics*, ASCE 111(8): 995–1009.
- Gray, R. J. 1984. Analysis of the effect of embedded fiber length on fiber debonding and pull-out from an elastic matrix. Part 2: Application to a steel fibre-cementitious matrix composite system, *Journal of Materials Science* 19: 1680–1691.
- Hannant, D. J. 1978. *Fibre Cements and Fibre Concretes*, John Wiley & Sons, Chichester.
- Hull, D. and Clyne, T. W. 1996. *An Introduction to Composite Materials*, Cambridge Solid State Science Series, second edition, Cambridge University Press, Cambridge.
- Karihaloo, B. L. and Fu, D. 1990a. An anisotropic damage model for plain concrete, *Engineering Fracture Mechanics* 35(1/2/3): 205–209.
- Karihaloo, B. L. and Fu, D. 1990b. An orthotropic damage model for plain concrete in tension, *ACI Materials Journal* 87(1): 62–67.
- Kimber, A. C. and Keer, J. G. 1982. On the theoretical average crack spacing in brittle matrix composites containing continuous aligned fibers, *Journal of Materials Science Letters* 1: 353–354.
- Krajcinovic, D. and Silva, M. A. G. 1982. Statistical aspects of the continuous damage theory, *International Journal of Solids and Structures* 18(7): 551–562.

- Li, F. 1998. *Fracture Characterization of Fiber Reinforced Concrete in Direct Uniaxial Tension*, Ph. D dissertation, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
- Li, F. and Li, Z. 1999. Acoustic emission from fiber reinforced concrete in direct tension. submitted to *ACI Materials Journal*.
- Li, S. H., Li, Z., Mura, T. and Shah, S. P. 1992. Multiple fracture of fiber reinforced brittle matrix composites based on micromechanics, *Engineering Fracture Mechanics* 43(4): 561–579.
- Li, V. C. and Leung, C. K. Y. 1992. Theory of steady state and multiple cracking of random discontinuous fiber reinforced brittle matrix composites, *Journal of Engineering Mechanics*, ASCE 118(11): 2246–2264.
- Li, Z., Li, F., Chang, T.-Y. P. and Mai, Y.-W. 1998. Uniaxial tensile behavior of concrete reinforced with randomly distributed short fibers, *ACI Materials Journal* 95(5): 564–574.
- Li, Z., Mobasher, B. and Shah, S. P. 1991. Characterization of interfacial properties in fiber reinforced cementitious composites, *Journal of American Ceramic Society* 74(9): 2156–2164.
- Løland, K. E. 1980. Continuous damage model for load-response estimation of concrete, *Cement and Concrete Research* 10: 395–402.
- Mazars, J. and Pijaudier-Cabot, G. 1989. Continuum damage theory — application to concrete, *Journal of Engineering Mechanics*, ASCE 115(2): 420–428.
- Naaman, A. E. and Homrich, J. R. 1989. Tensile stress strain properties of SIFCON, *ACI Materials Journal* 86(3): 244–251.
- Pompo, A., Stupak, P. R., Nicolais, L. and Marchese, B. 1996. Analysis of steel fibre pull-out from a cement matrix using video photography, *Cement and Concrete Composites* 18(1): 3–8.
- Shah, S. P., Swartz, S. E. and Ouyang, C. 1995. *Fracture Mechanics of Concrete: Applications of Fracture Mechanics to Concrete, Rock, and Other Quasi-Brittle Materials*, John Wiley & Sons, Inc., New York.

- Stang, H., Li, Z. and Shah, S. P. 1990. Pull-out problem: Stress versus fracture mechanical approach, *Journal of Engineering Mechanics*, ASCE 116(10): 2136–2150.
- Stang, H., Tonnesen, M. and Byskov, E. 1990. CDM based constitutive equations for high strength and fiber reinforced concrete, in S. P. Shah, S. E. Swartz and M. L. Wang (eds), *Micromechanics of Failure of Quasi-Brittle Materials*, Proceedings of the International Conference on Micromechanics of Failure of Quasi-Brittle Materials, Albuquerque, New Mexico, USA, Elsevier Applied Science, London, pp. 589–599.
- Zollo, R. F. 1997. Fiber reinforced concrete: an overview after 30 years of development, *Cement and Concrete Composites* 19: 107–122.

Appendix 1

Notations

A	(m ²)	= cross-sectional area of the fiber
A_c	(m ²)	= initial cross-sectional area of the FRC specimen
A_m	(m ²)	= initial cross-sectional area of the matrix
a	(m)	= debonded length of the fiber-matrix interface
B_1	(—)	= a coefficient defined by Eq. (11)
B_2	(—)	= a coefficient defined by Eq. (11)
D	(—)	= a dimensionless constant defined by Eq. (16)
D_{max}	(m)	= extension of the fracture zone in a matrix concrete
d_f	(m)	= diameter of the fiber
E_c	(Pa)	= elastic modulus of the composite
E_f	(Pa)	= elastic modulus of the fiber
E_m	(Pa)	= elastic modulus of the matrix
E_{m0}	(Pa)	= elastic modulus of the uncracked matrix
F	(N)	= externally applied force
f_i	(N)	= force in composite bar
f_m	(N)	= force in the matrix at which matrix damage is initiated
f_{mt}	(N/m)	= tensile strength of the matrix
g	(—)	= a constant related to fiber type and fiber content, appeared in Eq. (40)
K	(—)	= a dimensionless constant related to fiber type
k	(N/m ²)	= fiber-matrix interface shear stiffness
k_c	(N/m)	= overall stiffness of the composite cross-section
k_d	(N/m)	= stiffness of the damaged composite bar
k'_f	(N/m)	= equivalent stiffness of the fiber in an FRC
k_m	(N/m)	= stiffness of the damaged matrix
k_{m0}	(N/m)	= stiffness of the uncracked matrix
L	(m)	= average crack spacing for composite showing multiple cracking response, or length of the specimen for composite showing single cracking response
ℓ	(m)	= length of the fiber
ℓ_c	(m)	= critical length of the fiber
ℓ_e	(m)	= embedment length of the fiber
$\bar{\ell}_e$	(m)	= mean embedment length of the fiber

ℓ_s	(m)	=	average crack spacing of the composite
M	(-)	=	total number of fibers in each of N composite bars
m	(-)	=	number of damaged fibers
N	(-)	=	number of the composite bars
n	(-)	=	number of composite bars in which the matrix has damaged
n_r	(-)	=	a dimensionless constant defined by Eq. (30)
P	(N)	=	force in the fiber
$\bar{P}_f$	(N)	=	average force in the fiber
P^*	(N)	=	pull-out force at the end of a fiber
q	(N/m)	=	shear force per unit fiber length of the fiber-matrix interface
q_f	(N/m)	=	constant frictional shear force of the fiber-matrix interface
q_F	(N/m)	=	maximum adhesional shear force of the fiber-matrix interface
R	(-)	=	Energy ratio between the specific energy consumed in micro-crack formation and that consumed in further cracking of the fracture zone
S	(-)	=	number of the matrix bars in each of n damaged composite bars
s	(-)	=	number of damaged matrix bars in each of P matrix bars
t	(-)	=	a dimensionless constant defined by Eq. (12)
U	(m)	=	deformation in the fiber
$\bar{U}_f$	(m)	=	average deformation in the fiber
U^*	(m)	=	fiber deformation at the pull-out end
V_f	(-)	=	volume fraction of fibers in the composite
V_m	(-)	=	volume fraction of matrix in the composite
x	(m)	=	a distance from the end of a fiber at the transverse matrix crack
z	(m)	=	overall displacement of the composite
α_1	(-)	=	coefficient representing the fraction of displacement due to the fibers
α_{1m}	(-)	=	α_1 corresponding to the strain capacity of the composite ε_{ca}
α_{10}	(-)	=	α_1 corresponding to a composite strain of ε_0^f at which fiber debonding is initiated
α_2	(-)	=	coefficient representing the fraction of displacement due to the matrix
α_{2m}	(-)	=	α_2 corresponding to the strain capacity of the composite ε_{ca}
β_f	(-)	=	stiffness ratio = k'_f/k_c
β_m	(-)	=	stiffness ratio = k_{m0}/k_c
ε	(-)	=	strain in the composite
ε_0	(-)	=	strain in the composite at which matrix damage is initiated

ε_0^f	(-)	= strain in the composite at which fiber debonding is initiated
ε_{ca}	(-)	= strain capacity of the FRC
ε_m	(-)	= strain in the matrix concrete
ε_t	(-)	= strain in the matrix concrete corresponding to the tensile strength
ε_u	(-)	= ultimate strain in the matrix concrete
ζ	(-)	= a dimensionless constant defined by Eq. (12)
η	(-)	= the length efficiency factor of a fiber
Γ	(m ⁻¹)	= critical energy release rate of the fiber-matrix interface
γ	(-)	= a coefficient defined by Eq. (28)
ν_m	(-)	= Poisson's ratio of the matrix concrete
ξ	(-)	= a dimensionless constant defined by Eq. (11)
σ	(Pa)	= nominal stress in the composite
σ_m	(Pa)	= nominal stress in the matrix concrete
σ_{fu}	(Pa)	= ultimate stress of the fiber
τ	(Pa)	= frictional shear stress of the fiber-matrix interface
τ_{fu}	(Pa)	= frictional bond strength of the fiber-matrix interface
ω_c	(-)	= measure of damage related to matrix concrete
ω_{ct}	(-)	= damage of matrix concrete corresponding to its tensile strength
ω_f	(-)	= measure of damage related to fibers
ω_{fc}	(-)	= critical damage value related to fibers

List of Tables

1	Summary of the properties of the fibers used in the study	26
2	Property parameters of the matrix concrete	27
3	Summary of the numerical results from the analytical model	28

List of Figures

1	Schematic diagram showing the parallel bar model.	29
2	Evolution of damage ω_c (a) and nominal stress (b) related to strain for concrete matrix in uniaxial tension.	30
3	Tensile stress-strain curves for FRC containing 6% Dramix type I steel fibers: comparison of the experimental results with model predicted.	31
4	Tensile stress-strain curves for FRC containing 7% Dramix type I steel fibers: comparison of the experimental results with model predicted.	32
5	Tensile stress-strain curves for FRC containing 8% Dramix type I steel fibers: comparison of the experimental results with model predicted.	33
6	Tensile stress-strain curves for FRC containing 3% Dramix type II steel fibers: comparison of the experimental results with model predicted.	34
7	Tensile stress-strain curves for FRC containing 3.5% Dramix type II steel fibers: comparison of the experimental results with model predicted.	35
8	Tensile stress-strain curves for FRC containing 4% Dramix type II steel fibers: comparison of the experimental results with model predicted.	36

Table 1 Summary of the properties of the fibers used in the study

Fiber name	Fiber type	ℓ (mm)	d_f (mm)	ℓ/d_f	Density (g/cm ³)	Tensile Strength (GPa)	Young's Modulus (GPa)
Dramix I	Steel	30	0.5	60	7.8	1.2	200
Dramix II	Steel	50	0.5	100	7.8	1.2	200

Note: ℓ = fiber length;
 d_f = fiber diameter;
 ℓ/d_f = fiber aspect ratio.

Table 2 Property parameters of the matrix concrete

Parameters	f_{mt} (MPa)	E_{m0} (GPa)	ϵ_t ($\mu m/m$)	ϵ_0 ($\mu m/m$)	$\epsilon_u^\dagger$ ($\mu m/m$)
average	3.97	28.86	148.30	48.80	406.3
standard deviation	0.17	2.40	7.8	5.1	45.8
coefficient of variation (%)	5	8	6	11	12

[†] based on Eq. (13)

Table 3 Summary of the numerical results from the analytical model

Parameters	Unit	Dramix I fiber (V_f)			Dramix II fiber (V_f)			Comments
		6%	7%	8%	3%	3.5%	4%	
β_f	—	0.0453	0.0443	0.0433	0.0664	0.0656	0.0649	Eq. (24)
β_m	—	0.8778	0.8588	0.8402	0.9357	0.9253	0.9151	Eq. (25)
E_c	GPa	30.903	31.251	31.600	29.919	30.097	30.276	Eq. (26)
ℓ_c	mm	78.95	78.95	78.95	78.95	78.95	78.95	Eq. (33)
L	mm	22.61	19.18	16.60	28.00	23.88	20.79	Eq. (32)
ω_{fc}	—	0.8024	0.8024	0.8024	0.8168	0.8168	0.8168	Eq. (34)
ε_{ca}	10^{-3}	3.970	4.564	5.158	3.856	4.431	5.006	Eq. (36)

Figure 1 Schematic diagram showing the parallel bar model.

Figure 2 Evolution of damage ω_c (a) and nominal stress (b) related to strain for concrete matrix in uniaxial tension.

Figure 3 Tensile stress-strain curves for FRC containing 6% Dramix type I steel fibers: comparison of the experimental results with model predicted.

Figure 4 Tensile stress-strain curves for FRC containing 7% Dramix type I steel fibers: comparison of the experimental results with model predicted.

Figure 5 Tensile stress-strain curves for FRC containing 8% Dramix type I steel fibers: comparison of the experimental results with model predicted.

Figure 6 Tensile stress-strain curves for FRC containing 3% Dramix type II steel fibers: comparison of the experimental results with model predicted.

Figure 7 Tensile stress-strain curves for FRC containing 3.5% Dramix type II steel fibers: comparison of the experimental results with model predicted.

Figure 8 Tensile stress-strain curves for FRC containing 4% Dramix type II steel fibers: comparison of the experimental results with model predicted.