The energy release rate and the J-integral of an electrically insulated crack in a piezoelectric material

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Abstract: In this work, we examine the energy release rate and the J-integral of an electrically insulated crack in a piezoelectric solid under remotely uniform electrical/mechanical loads. We model the crack as a slender elliptical flaw to take into account the electric field inside the crack and obtain the exact and explicit solution. The analytic results show that the energy release rate, in general, does not equal to the J-integral and that the J-integral is path-dependent due to the electric traction on the crack faces. However, the energy release rate is equal to the sum of the J-integral along an infinitesimal circle around the crack tip and the average driving force produced by the electric field on the entire crack surface. Under combined mechanical and electrical loading, the energy release rate may be positive or negative, depending on the level of the mechanical load and the strength of the electric field as well. The effect of electric field on the fracture behavior is mechanical load-dependent, which has been observed in experiments.

Keywords: Piezoelectric material, Electrically insulated crack, Energy release rate, J-integral.

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1. Introduction

Fracture of piezoelectric materials has been of considerable interest in recent decades. One can cite the work of Parton [1], Deeg [2], Pak [3], Sosa [4], Zhang and Hack [5], Suo et al. [6], Park and Sun [7], Gao and Barnett [8], Zhang and Tong [9], Zhang et al. [10], Ru [11], Gao and Fan [12], McMeeking [13, 14], and others. Recently, an extensive summary of the previous work was presented in [15]. However, it should be noted that nearly all previous works focus mainly on mathematical cracks without an initial crack opening. In fact, a real physical crack has a finite width. In 1989, McMeeking [16] modeled a crack in an isotropic dielectric material as a sharp elliptical flaw with a low permittivity. He found that the crack-tip field was dependent on the ratio: $(b/a)/(\varepsilon_f/\varepsilon_m)$, where ε_f and ε_m are the permittivities of the flaw and the matrix, respectively; and a and b are the major and minor semi-axes of the elliptical flaw, respectively. McMeeking [16] pointed out that neglecting the electric field inside the crack might lead to an erroneous estimate of crack-tip fields. Zhang and Tong [9], and Zhang et al. [10] further developed McMeeking's work [16] for piezoelectric media by introducing a similar ratio defined as $\lambda = (b/a)/(\varepsilon_f/\varepsilon_{eff})$, where ε_{eff} is an effective permittivity of the piezoelectric matrix. We call λ the electrical impermeability of the crack because the higher the ratio is, the more impermeable the crack will be. Typically, a crack can be classified into three types, i.e., permeable crack ($\lambda = 0$), semi-permeable crack ($\lambda =$ a nonzero finite value), and impermeable crack ($\lambda \rightarrow \infty$) [9, 10]. In fracture mechanics analysis, the permeable crack model is equivalent to letting $b = 0$, while the impermeable crack model is equivalent to taking $\varepsilon_f = 0$. In addition to the ratio λ , Zhang [17] introduced the ratio of $Z = (h/H)/(\varepsilon_f/\varepsilon_{eff})$, where h and H denote the sizes of the crack width and the specimen,

respectively. The results of a simple capacitor-model indicate that the ratio Z functions similarly to the ratio λ and influences the energy release rate. McMeeking [13] also analyzed the energy release rate with a capacitor crack model, in which a mechanical load opens the crack, and obtained the similar conclusion that the crack opening (width) plays an important role in the energy release rate. More recently, Li [18] modeled a mode-III crack as a rectangular slit with an infinitesimal width in a piezoelectric solid, and calculated the local and global energy release rates. McMeeking [19] further studied the energy release rate of a general mode crack, modeled as a sharp elliptical cavity, in an infinite piezoelectric material. The energy change due to the introduction of the cavity is determined by the Eshelby inclusion technique and then the energy release rate is calculated. The results [18,19] show that the initial opening of the crack may play an important role in crack propagation, which has been confirmed by a recent experimental observation [20].

In the present work we study a generalized 2D problem of a crack in a piezoelectric solid. The crack is treated as a slender elliptical flaw with a minor semi-axis $b = \delta_c$ and a major semi-axis a . We let $\delta_c \rightarrow 0$ while holding λ finite and nonzero. The present approach is similar to McMeeking's work [19], in which the traditional energy balance approach is used to derive an explicit expression for the energy release rate of an electrically insulated crack. However, we place our emphasis on the J-integral and on the relationship between the energy release rate and the J-integral. Below is the outline of the present paper: Section 2 gives the basic equations, and Section 3 lists the explicit results of an infinite piezoelectric solid with a sharp elliptical cavity with an electric field inside the cavity and complex potentials in the material. From the viewpoint of the traditional energy balance, the energy release rate G is derived in Section 4, and the relationship between the G and the J-integral is established in Section 5. As an application of the obtained results above, we

discuss in details the effect of electric loading on piezoelectric fracture in Section 6. Finally, Section 7 concludes the work.

2. Basic Equations

In a rectangular coordinate system x_i ($i = 1, 2, 3$), the basic equations for a linear piezoelectric solid are [21]

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \quad (1)$$

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\varphi_{,i}, \quad (2)$$

$$\sigma_{ij} = C_{ijkl}\gamma_{kl} - e_{kij}E_k, \quad D_k = e_{kij}\gamma_{ij} + \varepsilon_{kl}E_l, \quad (3)$$

where u_i , φ , σ_{ij} , γ_{ij} , D_j and E_i are the displacement, the electric potential, the stress, the strain, the electric displacement and the electric field, respectively, and C_{ijkl} , e_{ijk} and ε_{ij} denote the elastic, the piezoelectric and the dielectric constants, respectively. Consider a generalized two-dimensional problem, in which all field variables are independent of x_3 . We introduce a generalized displacement vector $\mathbf{u}$ as [15, 21]

$$\mathbf{u} = [u_1, u_2, u_3, \varphi]^T = \mathbf{a} f(x_1 + px_2), \quad (4)$$

where the superscript T represents the transpose, $f(x_1 + px_2)$ is an analytic function, p is a complex number, and $\mathbf{a}$ a constant four-element column vector. Equations (1), (2) and (3) can be satisfied by (4) for arbitrary $f(x_1 + px_2)$ if

$$\left[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T} \right] \mathbf{a} = \mathbf{0}, \quad (5)$$

where the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are given by

$$\mathbf{Q} = \begin{bmatrix} C_{i1k1} & e_{11i} \\ e_{11i}^T & -\varepsilon_{11} \end{bmatrix}, \mathbf{R} = \begin{bmatrix} C_{i1k2} & e_{21i} \\ e_{12i}^T & -\varepsilon_{12} \end{bmatrix}, \mathbf{T} = \begin{bmatrix} C_{i2k2} & e_{22i} \\ e_{22i}^T & -\varepsilon_{22} \end{bmatrix}, i, k = 1, 2, 3.$$

The condition for nontrivial solutions to (5) requires

$$\left| \mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T} \right| = 0. \quad (6)$$

It can be shown that (6) has eight roots p_α and $\bar{p}_\alpha$ ($\alpha = 1, 2, 3, 4$) where p_α cannot be real because of the positive definiteness of the strain energy and electric energy densities. It is convenient to calculate p_α by solving the following standard eigen-equation:

$$\mathbf{N}\zeta = p\zeta, \quad (7)$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{pmatrix}, \zeta = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix},$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^T, \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} = \mathbf{N}_3^T \text{ and } \mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a}.$$

The general solution of (1), (2) and (3) can be expressed as

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z_*) + \overline{\mathbf{A}}\overline{\mathbf{f}}(\overline{z_*}), \quad (8)$$

$$\boldsymbol{\phi} = \mathbf{B}\mathbf{f}(z_*) + \overline{\mathbf{B}}\overline{\mathbf{f}}(\overline{z_*}), \quad (9)$$

where

$$\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4), \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4),$$

$$\mathbf{f}(z_*) = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^T, z_\alpha = x_1 + p_\alpha x_2,$$

and $\boldsymbol{\phi}$ is the generalized stress function such that

$$\boldsymbol{\sigma}_1 = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1]^T = -\boldsymbol{\phi}_{,2}, \quad (10)$$

$$\boldsymbol{\sigma}_2 = [\sigma_{12}, \sigma_{22}, \sigma_{23}, D_2]^T = \boldsymbol{\phi}_{,1}. \quad (11)$$

3. Field Intensity Factors for a Semi-Permeable Crack

Consider an electrically insulated elliptical cavity of major semi-axis a and minor semi-axis b in an infinite piezoelectric medium subjected to the far-field mechanical/electric loading $\boldsymbol{\Sigma}_{2\infty} = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^T$ and $\boldsymbol{\Sigma}_{1\infty} = [\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{13}^\infty, D_1^\infty]^T$, as shown in Fig. 1. There is air or vacuum in the cavity with the dielectric constant ε_0 and the cavity surface is traction free. The solution in the cavity can be found in the literature, e.g., Zhang et al. [10], Gao and Fan [12], Gao [22], and others. Also according to these existing works, the complex function $\mathbf{f}(z_*)$ has the form

$$\mathbf{f}(z_*) = \langle\langle z_\alpha \rangle\rangle \mathbf{c}^\infty + \mathbf{f}_0(z_*), \quad (12)$$

where $\langle\langle \rangle\rangle$ denotes a diagonal matrix; $\mathbf{f}_0(z)$ is an unknown complex function, which is analytic at infinity, i.e., $\mathbf{f}_0(\infty) = \mathbf{0}$; $\mathbf{c}^\infty$ is a constant related to the field variables at infinity such that

$$\mathbf{c}^\infty = \mathbf{B}^T \boldsymbol{\varepsilon}_{1\infty} + \mathbf{A}^T \boldsymbol{\Sigma}_{2\infty},$$

with

$$\boldsymbol{\varepsilon}_{1\infty} = [\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + \omega_3^\infty, 2\varepsilon_{13}^\infty, -E_1^\infty]^T.$$

Substituting (12) into (8) and (9) leads to

$$\boldsymbol{\phi} = \boldsymbol{\Sigma}_{2\infty} x_1 - \boldsymbol{\Sigma}_{1\infty} x_2 + 2 \operatorname{Re}[\mathbf{K}_0(z_*)], \quad (13)$$

$$\mathbf{u} = \boldsymbol{\varepsilon}_{1\infty} x_1 + \boldsymbol{\varepsilon}_{2\infty} x_2 + 2 \operatorname{Im}[\mathbf{Y} \mathbf{K}_0(z_*)], \quad (14)$$

where

$$\boldsymbol{\varepsilon}_{2\infty} = [\varepsilon_{21}^\infty - \omega_3^\infty, \varepsilon_{22}^\infty, 2\varepsilon_{23}^\infty, -E_2^\infty]^T, \quad \mathbf{K}_0(z_*) = \mathbf{B}\mathbf{f}_0(z), \quad \mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1}.$$

In the equations above, the generalized strain vectors $\boldsymbol{\varepsilon}_{1\infty}$ and $\boldsymbol{\varepsilon}_{2\infty}$ can be expressed in terms of σ_{ij}^∞ and D_j^∞ through the constitutive equations. Generally, the additional condition of zero rotation around the x_3 -axis at infinity, i.e., $\omega_3^\infty = 0$, can be imposed.

Under remotely uniform loading, the electric displacement inside the cavity is uniform with components D_2^0 and D_1^0 . Thus, we can determine $\mathbf{K}_0(z_*)$ in (13) and (14) from the boundary condition on the cavity surface. The final result is [22]

$$\mathbf{K}_0(z_*) = -\frac{1}{2} \langle\langle \zeta_\alpha^{-1}(z_\alpha) \rangle\rangle \mathbf{p}, \quad (15)$$

where

$$\zeta_\alpha^{-1}(z_\alpha) = \frac{z_\alpha - \sqrt{z_\alpha^2 - c_\alpha^2}}{a + ip_\alpha b}, \quad c_\alpha^2 = a^2 + p_\alpha^2 b^2, \quad (16)$$

$$\mathbf{p} = [a\sigma_{21}^\infty - ib\sigma_{11}^\infty, a\sigma_{22}^\infty - ib\sigma_{12}^\infty, a\sigma_{23}^\infty - ib\sigma_{13}^\infty, a(D_2^\infty - D_2^0) - ib(D_1^\infty - D_1^0)]^T, \quad (17)$$

$$D_2^\infty - D_2^0 = \frac{b(D_2^\infty - \varepsilon_0 E_2^\infty) + \varepsilon_0 \operatorname{Re} \sum_{j=1}^3 Y_{4j} M_j^\infty}{b - a\varepsilon_0 Y_{44}}, \quad (18)$$

$$D_1^\infty - D_1^0 = \frac{a(D_1^\infty - \varepsilon_0 E_1^\infty) - \varepsilon_0 \operatorname{Im} \sum_{j=1}^3 Y_{4j} M_j^\infty}{a - b\varepsilon_0 Y_{44}}, \quad (19)$$

$$\mathbf{M}^\infty = [a\sigma_{21}^\infty - ib\sigma_{11}^\infty, a\sigma_{22}^\infty - ib\sigma_{12}^\infty, a\sigma_{23}^\infty - ib\sigma_{13}^\infty]^T. \quad (20)$$

For a slender elliptical flaw with $b = \delta_c \rightarrow 0$, $b\varepsilon_0$ is so small that all terms related to $b\varepsilon_0$ may be neglected in (18) and (19), which then can be approximated as

$$D_2^\infty - D_2^0 = \frac{bD_2^\infty + a \operatorname{Re} \sum_{j=1}^3 Y_{4j} \sigma_{2j}^\infty}{b - a\varepsilon_0 Y_{44}}, \quad (21)$$

$$D_1^\infty - D_1^0 = D_1^\infty - \varepsilon_0 E_1^\infty - \varepsilon_0 \operatorname{Im} \sum_{j=1}^3 Y_{4j} \sigma_{2j}^\infty. \quad (22)$$

From (21) and (22), we can give the electric field inside the crack as

$$D_2^0 = D_2^\infty - \frac{\lambda D_2^\infty - \frac{1}{H_{44}} \sum_{j=1}^3 H_{4j} \sigma_{2j}^\infty}{1 + \lambda}, \quad (23)$$

$$D_1^0 = \varepsilon_0 E_1^\infty + \varepsilon_0 \sum_{j=1}^3 \operatorname{Im}[Y_{4j}] \sigma_{2j}^\infty, \quad (24)$$

where $\mathbf{H} = \mathbf{Y} + \overline{\mathbf{Y}}$, and

$$\lambda = \alpha / \beta, \quad \alpha = \delta_c / a, \quad \beta = \varepsilon_0 / \varepsilon^{\text{eff}}, \quad \varepsilon^{\text{eff}} = -Y_{44}^{-1}.$$

For this case, the complex potential is still given by (15) with the simplified expressions of $\mathbf{p}$ and

$\zeta_\alpha^{-1}(z_\alpha)$:

$$\mathbf{p}_c = a\Pi_{2\infty}, \quad \Pi_{2\infty} = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty - D_2^0]^T, \quad (25)$$

$$\zeta_\alpha^{-1}(z_\alpha) = \frac{1}{a} \left(z_\alpha - \sqrt{z_\alpha^2 - a^2} \right). \quad (26)$$

Then, the intensity factor vector takes the form

$$\mathbf{k} = \sqrt{\pi a} \Pi_{2\infty}, \quad (27)$$

where $D_2^\infty - D_2^0$ in $\Pi_{2\infty}$ is determined from (23).

4. Energy Release Rate

From the thermodynamic point of view, the energy release rate is the most appropriate physical quantity to characterize the fracture behavior. For purely elastic materials, the energy release rate G is identical to the J-integral because there is no energy stored in the crack cavity. For an electrically insulated crack in a piezoelectric material, however, a high electric field exists inside the crack. This electric energy can change the relationship between G and J . In this section, we shall derive the exact expression of the energy release rate based on the principle of isothermal piezoelectric virtual work [10, 15].

There are four isothermal thermodynamic functions: the free energy, f , the electric enthalpy, h , the mechanical enthalpy, w , and the full Gibbs free energy, g [10, 15]. Each of the four isothermal thermodynamic functions corresponds to an isothermal potential, P , which is defined as the difference between the generalized work, W , done by mechanical and electrical loads and the generalized mechanical and electrical energy, U , stored inside the body of interest. In the differential form, we have energy balance condition:

$$\partial P = \partial W - \partial U . \quad (28)$$

Using the electric enthalpy h , the generalized work and the generalized stored energy are given by

$$W = \int_{\Gamma} (\sigma_{ij} n_j u_i + D_i n_i \varphi) d\Gamma , \quad (29)$$

$$U = \iiint_{\Omega} h d\Omega , \quad (30)$$

respectively, where Γ denotes a contour as shown in Fig. 2, Ω is the integration domain enclosed by Γ , u_i is the mechanical displacement, $D_i n_i$ is the prescribed electric displacement on Γ , and the electric enthalpy per unit volume h takes the form:

$$h = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} - \frac{1}{2} D_i E_i. \quad (31)$$

For 2D cracks, the energy release rate, G , for crack propagation can be defined as

$$G = \frac{\partial P}{\partial a} = \frac{\partial}{\partial a} (W - U). \quad (32)$$

On the other hand, one has

$$U = \iint_{\Omega} h d\Omega = \iint_m h d\Omega + \iint_c h d\Omega, \quad (33)$$

where m and c stand for the material and the cavity, respectively. Using Green's theorem, we have

$$\iint_m h d\Omega = \int_{\Gamma} \frac{1}{2} (\sigma_{ij} n_j u_i + D_i n_i \varphi) d\Gamma + \oint_{\Gamma'} \frac{1}{2} (D_i n_i \varphi) d\Gamma', \quad (34)$$

$$\iint_c h d\Omega = - \oint_{\Gamma'} \frac{1}{2} (D_i n_i \varphi) d\Gamma'. \quad (35)$$

where Γ' is the contour of the cavity as shown in Fig. 2. Substituting (34) and (35) in (33) results in

$$U = \frac{1}{2} \int_{\Gamma} (\sigma_{ij} n_j u_i + D_i n_i \varphi) d\Gamma. \quad (36)$$

Using (29) and (36), one obtains

$$W - U = \frac{1}{2} \int_{\Gamma} (\sigma_{ij} n_j u_i + D_i n_i \varphi) d\Gamma. \quad (37)$$

Substituting (37) into (32) gives

$$G = \frac{1}{2} \int_{\Gamma} \left(\sigma_{ij} n_j \frac{\partial u_i}{\partial a} + D_i n_i \frac{\partial \varphi}{\partial a} \right) d\Gamma. \quad (38)$$

Noting that $n_3 = 0$ on Γ , we have

$$\begin{aligned}
\sigma_{ij}n_j \frac{\partial u_i}{\partial a} + D_i n_i \frac{\partial \varphi}{\partial a} &= \left(\sigma_{11} \frac{\partial u_1}{\partial a} + \sigma_{21} \frac{\partial u_2}{\partial a} + \sigma_{31} \frac{\partial u_2}{\partial a} + D_1 \frac{\partial \varphi}{\partial a} \right) n_1 \\
&+ \left(\sigma_{12} \frac{\partial u_1}{\partial a} + \sigma_{22} \frac{\partial u_2}{\partial a} + \sigma_{32} \frac{\partial u_2}{\partial a} + D_2 \frac{\partial \varphi}{\partial a} \right) n_2 = \boldsymbol{\sigma}_1^T \frac{\partial \mathbf{u}}{\partial a} n_1 + \boldsymbol{\sigma}_2^T \frac{\partial \mathbf{u}}{\partial a} n_2. \quad (39)
\end{aligned}$$

Using (10) and (11) in (39) gives

$$\sigma_{ij}n_j \frac{\partial u_i}{\partial a} + D_i n_i \frac{\partial \varphi}{\partial a} = -\boldsymbol{\phi}_{,2}^T \frac{\partial \mathbf{u}}{\partial a} n_1 + \boldsymbol{\phi}_{,1}^T \frac{\partial \mathbf{u}}{\partial a} n_2. \quad (40)$$

Due to

$$n_1 d\Gamma = dx_2, \quad n_2 d\Gamma = -dx_1, \quad (41)$$

we obtain from (40) and (41) that

$$\begin{aligned}
\left(\sigma_{ij}n_j \frac{\partial u_i}{\partial a} + D_i n_i \frac{\partial \varphi}{\partial a} \right) d\Gamma &= -\left[\boldsymbol{\phi}_{,2}^T dx_2 + \boldsymbol{\phi}_{,1}^T dx_1 \right] \cdot \frac{\partial \mathbf{u}}{\partial a}, \\
&= -\left[d\boldsymbol{\phi}^T \right] \cdot \frac{\partial \mathbf{u}}{\partial a}. \quad (42)
\end{aligned}$$

Inserting (42) into (38) leads to

$$G = -\frac{1}{2} \int_{\Gamma} \left[d\boldsymbol{\phi}^T \right] \cdot \frac{\partial \mathbf{u}}{\partial a}. \quad (43)$$

From (14) one has

$$\frac{\partial \mathbf{u}}{\partial a} = 2 \operatorname{Im} \left[\mathbf{Y} \frac{\partial}{\partial a} \mathbf{K}_0(z_*) \right]. \quad (44)$$

If we take Γ as a large elliptical contour enclosing the cavity, then

$$x_1 = a\rho_0 \cos \theta, \quad x_2 = \delta_c \rho_0 \sin \theta, \quad \rho_0 \gg 1, \quad (45)$$

with $\rho \bar{\rho} = \rho_0^2$ and $\zeta_\alpha = \rho = \rho_0 e^{i\theta}$ on Γ . One can write (45) as

$$x_1 = \frac{1}{2}a\left(\frac{\rho_0^2}{\rho} + \rho\right), \quad x_2 = \frac{i}{2}\delta_c\left(\frac{\rho_0^2}{\rho} - \rho\right), \quad \rho = \rho_0 e^{i\theta}, \quad \text{on } \Gamma. \quad (46)$$

Moreover, one has from (15) that

$$\mathbf{K}_0(z_*) = -\frac{1}{2\rho}\mathbf{p}_c, \quad \text{on } \Gamma. \quad (47)$$

Substituting (47) into (44) leads to

$$\frac{\partial \mathbf{u}}{\partial a} = -\text{Im}\left[\mathbf{Y} \frac{\partial \mathbf{p}_c}{\partial a} \frac{1}{\rho}\right]. \quad (48)$$

Equation (48) can be rewritten as

$$\frac{\partial \mathbf{u}}{\partial a} = \mathbf{U}_1 \rho + \mathbf{U}_{-1} \frac{1}{\rho}, \quad (49)$$

where

$$\mathbf{U}_1 = \frac{1}{2\rho_0^2} i\bar{\mathbf{Y}} \frac{\partial \bar{\mathbf{p}}_c}{\partial a}, \quad \mathbf{U}_{-1} = \frac{1}{2} i\mathbf{Y} \frac{\partial \mathbf{p}_c}{\partial a}. \quad (50)$$

A substitution of (46) and (47) into (13) yields

$$\boldsymbol{\phi} = \mathbf{W}_1 \rho + \mathbf{W}_{-1} \rho^{-1}, \quad (51)$$

where

$$\mathbf{W}_1 = \frac{1}{2} \left(a\boldsymbol{\Sigma}_{2\infty} + i\delta_c \boldsymbol{\Sigma}_{1\infty} - \bar{\mathbf{p}}_c / \rho_0^2 \right), \quad (52)$$

$$\mathbf{W}_{-1} = \frac{1}{2} \left(a\boldsymbol{\Sigma}_{2\infty} \rho_0^2 - i\delta_c \boldsymbol{\Sigma}_{1\infty} \rho_0^2 - \mathbf{p}_c \right). \quad (53)$$

From (51) one has

$$d\boldsymbol{\phi}^T = (\mathbf{W}_1^T - \mathbf{W}_{-1}^T \rho^{-2}) d\rho. \quad (54)$$

Using (54) and (49), we obtain

$$\left[d\boldsymbol{\phi}^T \right] \cdot \frac{\partial \mathbf{u}}{\partial a} = \left(\mathbf{W}_1^T - \mathbf{W}_{-1}^T \rho^{-2} \right) \left(\mathbf{U}_1 \rho + \mathbf{U}_{-1} \rho^{-1} \right) d\rho, \quad (55)$$

Substituting (55) into (43) leads to

$$G = -\pi i \left(\mathbf{W}_1^T \mathbf{U}_{-1} - \mathbf{W}_{-1}^T \mathbf{U}_1 \right). \quad (56)$$

Finally, substituting (50), (52) and (53) into (56), and neglecting the terms of the order of ρ_0^{-2} as

$\rho_0 \rightarrow \infty$, we find

$$G = \frac{\pi a}{2} \text{Im} \left[\left(\Sigma_{2\infty}^T + i \Sigma_{1\infty}^T \delta_c / a \right) i \mathbf{Y} \frac{\partial \mathbf{p}_c}{\partial a} \right]. \quad (57)$$

For $\delta_c / a \rightarrow 0$, (57) becomes

$$G = \frac{\pi a}{2} \text{Im} \left[\Sigma_{2\infty}^T i \mathbf{Y} \frac{\partial \mathbf{p}_c}{\partial a} \right], \quad (58)$$

The term $\frac{\partial \mathbf{p}_c}{\partial a}$ can be determined from the first equation of (25). In the present work, we assume

that the crack extends in a self-similar manner, that is, $b/a = \text{a constant}$, which was adopted previously by Zhang et al. [10] and McMeeking [19]. It can be shown that this condition is equivalent to $D_2^0 = \text{a constant}$ during crack propagation. With this assumption, we have

$$\frac{\partial \mathbf{p}_c}{\partial a} = \left[\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty - D_2^0 \right]^T = \Pi_{2\infty}. \quad (59)$$

Substituting (59) into (58), we obtain the expression for the energy release rate of a semi-permeable crack as

$$G = \frac{\pi a}{4} \Sigma_{2\infty}^T \mathbf{H} \Pi_{2\infty}. \quad (60)$$

Since $\Sigma_{2\infty} = \Pi_{2\infty} + \mathbf{i}_4 D_2^0$, where $\mathbf{i}_4 = (0, 0, 0, 1)^T$, (60) can be rewritten as

$$G = \frac{\pi a}{4} \left(\Pi_{2\infty}^T + \mathbf{i}_4^T D_2^0 \right) \mathbf{H} \Pi_{2\infty},$$

that is

$$G = \frac{\pi a}{4} \Pi_{2\infty}^T \mathbf{H} \Pi_{2\infty} + \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (61)$$

Equation (61) can be further reduced to the form

$$G = \frac{1}{4} \mathbf{k}^T \mathbf{H} \mathbf{k} + \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0, \quad (62)$$

where $\mathbf{k}$ is the stress intensity vector defined in (27). Using the Eshelby inclusion technique, McMeeking [19] recently derived an expression of the energy release rate similar to (62). His equation (27) is identical (23) in the present work. However, there is a $\varepsilon_0 E_2^\infty$ term in the first right term of his equation (26), which is neglected in his equation (27) and in the present work. Except of the $\varepsilon_0 E_2^\infty$ term, McMeeking's result is the same as (62).

5. Relationship between G and the J-integral

The J-integral in a piezoelectric solid can be defined as [23]

$$J = \int_{\Gamma_*} (h n_1 - \sigma_{ij} n_j u_{i,1} - n_i D_i \varphi_{,1}) ds, \quad (63)$$

where Γ_* is a contour enclosing the right crack tip, starting from a point on the lower crack surface and terminating at a point on the upper crack surface. Let us consider the J-integral of a closed contour $\Gamma_0 = \Gamma_\gamma + (-\Gamma_l) + (-\Gamma_{cs})$, where Γ_γ is any contour integral starting at a point x_1^- on the lower crack surface and ending at the corresponding point x_1^+ on the upper crack surface, Γ_l is an infinitesimal circle of radius r_0 around the crack tip, and Γ_{cs} represents parts of crack surfaces AB and BC as shown in Fig. 3. Since the J-integral for a closed contour is zero, we have

$$J_0 = J_\gamma - J_l - \hat{J}_{cs} = 0, \quad (64)$$

where J_l may be called the local energy release rate along Γ_l , J_γ is the J-integral along the contour Γ_γ , and

$$\hat{J}_{cs} = J_{AB} + J_{BC} \quad (65)$$

is the contribution from the electrical field along the crack surfaces AB and BC. Using (63), we have

$$J_{AB} = \int_{AB} (h + D_1 E_1) n_1 ds - \int_{AB} D_2^0 \varphi_{,1} n_2 ds. \quad (66)$$

On the crack surface, $h + D_1 E_1$ is free of singularities and $n_1 \approx 0$ because $b = \delta_c \rightarrow 0$ and thus, the first integral in (66) can be neglected and (66) reduces to

$$J_{AB} = D_2^0 [\varphi^-(a^*) - \varphi^-(x_1)], \quad (67)$$

where $a^* = a - r_0$. Similarly, on the lower crack surface BA, one has

$$J_{BC} = D_2^0 [\varphi^+(x_1) - \varphi^+(a^*)]. \quad (68)$$

Using (65), (67) and (68), we have

$$\hat{J}_{cs}(x_1) = J_{AB} + J_{BC} = D_2^0 [\varphi^-(a^*) - \varphi^-(x_1)] + D_2^0 [\varphi^+(x_1) - \varphi^+(a^*)]. \quad (69)$$

For $r_0 \rightarrow 0$, $a^* \rightarrow a$ and $\varphi^+(a^*) \rightarrow \varphi^-(a^*)$, (69) approaches

$$\hat{J}_{cs}(x_1) = D_2^0 [\varphi^+(x_1) - \varphi^-(x_1)]. \quad (70)$$

We can use the crack opening (14) to calculate the potential difference:

$$\varphi^+(x_1, 0) - \varphi^-(x_1, 0) = 2 \operatorname{Im} \langle \mathbf{Y} [\mathbf{K}_0^+(x_1, 0) - \mathbf{K}_0^-(x_1, 0)] \rangle_4, \quad (71)$$

where $\langle \rangle_4$ denotes the fourth row of the vector in $\langle \rangle$. Substituting (15) together with (25) and (26) into (71) yields

$$\varphi^+(x_1, 0) - \varphi^-(x_1, 0) = 2 \operatorname{Im} \langle \mathbf{Y} i \Pi_{2\infty} \rangle_4 \sqrt{a^2 - x_1^2} = \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} \sqrt{a^2 - x_1^2}, \quad x_1 \in [-a, a]. \quad (72)$$

Thus, (70) becomes

$$\hat{J}_{cs}(x_1) = \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0 \sqrt{a^2 - x_1^2}, \quad x_1 \in [-a, a]. \quad (73)$$

Since $\hat{J}_{cs}(x_1)$ does not, in general, equal to zero, (64) indicates $J_\gamma - J_l = \hat{J}_{cs}(x_1) \neq 0$, and thereby shows that the integral J_γ is path-dependent for an electrically insulated crack.

On the other hand, it can be shown from (63) that

$$J_l = \frac{\pi a}{4} \Pi_{2\infty}^T \mathbf{H} \Pi_{2\infty} = \frac{1}{4} \mathbf{k}^T \mathbf{H} \mathbf{k}. \quad (74)$$

Then, (62) can be rewritten as

$$G = J_l + \Delta G, \quad (75)$$

where J_l is the local energy release rate and

$$\Delta G = \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (76)$$

To illustrate the physical meaning of ΔG , we introduce the parameter J_{cs} defined as

$$J_{cs} = \frac{1}{a} \int_0^a \hat{J}_{cs}(x_1) dx_1, \quad (77)$$

which represents an average driving force produced by the electric field of the entire crack surfaces, as shown in Fig. 4. Substituting (73) into (77) and carrying out the integration, we find

$$J_{cs} = \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (78)$$

Comparing (78) and (76) gives

$$\Delta G = J_{cs}. \quad (79)$$

i.e., ΔG is the average contribution of the J-integral along the crack faces. This contribution is from the electric field within the crack because J_{cs} involves only D_2^0 . Substituting (79) into (75) gives

$$G = J_l + J_{cs}. \quad (80)$$

Equation (80) gives the relationship between the global energy release rate G and the J-integrals, i.e., the global energy release rate equals to the sum of the local energy release rate J_l and the average contribution of the J-integral along the crack faces. In other words, the J-integral J_l is not, in general, equal to G for an electrically insulated crack since $J_{cs} \neq 0$ and moreover the J-integral J_γ is path-dependent.

Now, let us examine the J-integral along Γ_g , which starts at $x_1^- = 0$ on the lower crack surface and ends at $x_1^+ = 0$ on the upper crack surface as shown in Fig. 3. Obviously the value of the J-integral along Γ_g can be determined from (64) as

$$J_g = J_l + \hat{J}(0), \quad (81)$$

where

$$\hat{J}(0) = \hat{J}_{cs}(0) = a\mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (82)$$

A substitution of (82) into (81) results in

$$J_g = J_l + a\mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (83)$$

Comparing (83) with (75) one finds $J_g \neq G$. This result is different from that of Li [18] who concluded that $J_g = G$ for a mode-III crack. This discrepancy might be resulted from the difference in the crack model. In Li's work, a crack is treated as a rectangular slit, while in the present work, a narrow elliptical cavity. In the present case, (83) and (75) give

$$J_g - G = 0.215 a \mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} D_2^0. \quad (84)$$

There are two extreme limits for an electrically insulated crack, i.e., electrically impermeable and permeable. For an impermeable crack, there is no electrical field inside the crack, i.e., $D_2^0 = 0$.

Thus, $\hat{J}_{cs}(x_1) = J_{cs} = 0$ and Eq. (80) reduces to

$$G = J_l = \frac{1}{4} \mathbf{k}_{im}^T \mathbf{H} \mathbf{k}_{im}. \quad (85)$$

where

$$\mathbf{k}_{im} = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^T \sqrt{\pi a}, \quad (86)$$

is the intensity factor vector of the impermeable crack. In this case, the energy release rate equals to the J integral and the J integral is path-independent.

For a permeable crack, $\lambda = 0$ and (23) results in

$$D_2^\infty - D_2^0 = -\frac{1}{H_{44}} \sum_{j=1}^3 H_{4j} \sigma_{2j}^\infty, \quad (87)$$

or

$$\sum_{j=1}^3 H_{4j} \sigma_{2j}^\infty + H_{44} (D_2^\infty - D_2^0) = 0. \quad (88)$$

From the second equation of (25), Eq. (88) implies

$$\mathbf{i}_4^T \mathbf{H} \Pi_{2\infty} = 0. \quad (89)$$

Substituting (89) into (73) and (78) yields

$$\hat{J}_{cs}(x_1) = J_{cs}(x_1) = 0. \quad (90)$$

Thus, (80) becomes

$$G = J_l = \frac{1}{4} \mathbf{k}_p^T \mathbf{H} \mathbf{k}_p, \quad (91)$$

where $\mathbf{k}_p$ is the intensity factor vector of the permeable crack given by

$$\mathbf{k}_p = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty - D_2^0]^T \sqrt{\pi a}, \quad (92)$$

in which $D_2^\infty - D_2^0$ is determined from (87). Equation (91) shows that for a permeable crack, the energy release rate also equals to the J integral and thus J is also path-independent.

6. Effect of Electric Fields on Piezoelectric Fracture

6.1 Under purely electric loading of D_2^∞ only

In this case, the second equations of (25) and (23) give

$$\Pi_{2^\infty} = \mathbf{i}_4 (D_2^\infty - D_2^0), \quad D_2^\infty - D_2^0 = \frac{\lambda}{1 + \lambda} D_2^\infty, \quad (93)$$

respectively. Equations (74) and (78) can be written as

$$\begin{aligned} J_l &= \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \mathbf{i}_4 (D_2^\infty - D_2^0)^2, \\ &= \frac{\pi a}{4} H_{44} \frac{\lambda^2}{(1 + \lambda)^2} (D_2^\infty)^2, \end{aligned} \quad (94)$$

$$\begin{aligned} J_{cs} &= \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} \mathbf{i}_4 (D_2^\infty - D_2^0) D_2^0, \\ &= \frac{\pi a}{4} H_{44} \frac{\lambda}{(1 + \lambda)^2} (D_2^\infty)^2. \end{aligned} \quad (95)$$

Substituting (94) and (95) into (80) results in

$$G = \frac{\pi a}{4} H_{44} \frac{\lambda}{1 + \lambda} (D_2^\infty)^2. \quad (96)$$

Since $H_{44} < 0$, one has $G < 0$, which implies that the crack always tends to close when an electric load is solely applied.

For a permeable crack, $\lambda = 0$,

$$G = 0; \quad (97)$$

and for an impermeable crack, $\lambda \rightarrow \infty$, G becomes

$$G = \frac{\pi a}{4} H_{44} (D_2^\infty)^2 < 0. \quad (98)$$

Note that (97) and (98) are the results from linear electro-elasticity, which predict that an electric load alone cannot propagate an electrically insulated crack.

The value of G from (96) is closer to that of (97) for $0 \leq \lambda < 1$, and hence the crack behaves more like a permeable slit. When $\lambda > 1$, G is closer to that of (98), which means that the crack behaves more like an impermeable slit. The variation of G with λ is shown in Fig. 5. Since δ_0 / a is small compared to $-Y_{44}\varepsilon_0$ for most of crack problems in engineering, which gives $\lambda < 1$, the permeable crack model may be more appropriate than the impermeable one. On the other hand, the impermeable boundary conditions are more appropriate for electrically insulated cavities, when $\lambda > 1$. Also from (94) and (95), under purely electrical loading, J_{cs} predominates in the global energy release rate when $\lambda < 1$, while J_l predominates for $\lambda > 1$.

6.2 Under combined mechanical and electrical loading of σ_2^∞ and D_2^∞

In this case, we have from the second equations of (25) and (23) that

$$\Pi_{2\infty} = \mathbf{i}_2 \sigma_2^\infty + \mathbf{i}_4 (D_2^\infty - D_2^0), \quad D_2^\infty - D_2^0 = \frac{\lambda}{1+\lambda} D_2^\infty - \frac{1}{1+\lambda} \frac{H_{42}}{H_{44}} \sigma_2^\infty, \quad (99)$$

respectively. Equation (61) can be written as

$$G = \frac{\pi a}{4} [\mathbf{i}_2 \sigma_2^\infty + \mathbf{i}_4 (D_2^\infty - D_2^0)]^T \mathbf{H} [\mathbf{i}_2 \sigma_2^\infty + \mathbf{i}_4 (D_2^\infty - D_2^0)] + \frac{\pi a}{4} \mathbf{i}_4^T \mathbf{H} [\mathbf{i}_2 \sigma_2^\infty + \mathbf{i}_4 (D_2^\infty - D_2^0)] D_2^0, \quad (100)$$

where $\mathbf{i}_2 = (0, 1, 0, 0)^T$. A substitution of (99) into (100) yields

$$G = \frac{\pi a}{4} \hat{H}_{22} (\sigma_2^\infty)^2 + \frac{\pi a}{4} H_{44} \frac{\lambda}{1+\lambda} (D_2^\infty)^2 + \frac{\pi a}{2} H_{42} \frac{\lambda}{1+\lambda} D_2^\infty \sigma_2^\infty. \quad (101)$$

where

$$\hat{H}_{22} = H_{22} - \frac{H_{42}^2}{H_{44}} > 0.$$

Equation (101) can be rewritten as

$$\hat{G} = \hat{H}_{22} (\sigma_2^\infty)^2 + \hat{\lambda} H_{44} (D_2^\infty)^2 + 2\hat{\lambda} H_{24} D_2^\infty \sigma_2^\infty, \quad (102)$$

where

$$\hat{G} = \frac{4G}{\pi a}, \quad \hat{\lambda} = \frac{\lambda}{1+\lambda}.$$

Form (102) and $\hat{H}_{22} > 0$, we have that for a given value of D_2^∞ , $\hat{G}$ is positive if the mechanical load σ_2^∞ exceeds a critical value

$$\sigma_{2c}^\infty = \frac{D_2^\infty}{\hat{H}_{22}} \left[\sqrt{(\hat{\lambda} H_{24})^2 + \hat{\lambda} \hat{H}_{22} (-H_{44})} - \hat{\lambda} H_{24} \right]. \quad (103)$$

When σ_2^∞ is below the critical value, $\hat{G}$ is negative. Equation (103) shows that for permeable cracks, $\hat{\lambda} = 0$, thus $\sigma_{2c}^\infty = 0$; and for impermeable cracks $\hat{\lambda} \rightarrow 1$, the critical mechanical load is

$$\sigma_{2c}^\infty = \frac{D_2^\infty}{\hat{H}_{22}} \left[\sqrt{(H_{24})^2 + \hat{H}_{22}(-H_{44})} - H_{24} \right]. \quad (104)$$

For a given value of σ_2^∞ , the variation of the energy release rate vs. the electric loading is shown in Fig. 6 for different values of H_{24} , where $[D_2^\infty]_I$, $[D_2^\infty]_2$, $[D_2^\infty]_0$ and d_0 are, respectively, given by

$$\begin{aligned} [D_2^\infty]_I &= \frac{\sigma_2^\infty}{\hat{\lambda}(-H_{44})} \left[\hat{\lambda}H_{24} - \sqrt{(\hat{\lambda}H_{24})^2 + \hat{\lambda}\hat{H}_{22}(-H_{44})} \right], \\ [D_2^\infty]_2 &= \frac{\sigma_2^\infty}{\hat{\lambda}(-H_{44})} \left[\hat{\lambda}H_{24} + \sqrt{(\hat{\lambda}H_{24})^2 + \hat{\lambda}\hat{H}_{22}(-H_{44})} \right], \\ [D_2^\infty]_0 &= \sigma_2^\infty \sqrt{\frac{\hat{H}_{22}}{-H_{44}}} \hat{\lambda}, \quad d_0 = \frac{H_{24}}{-H_{44}} \sigma_2^\infty. \end{aligned} \quad (105)$$

For a poled ceramic with the poling direction being along the positive (or negative) x_2 axis, $H_{24} > 0$ (or $H_{24} < 0$), while $H_{24} = 0$ for a depoled ceramic. The effects of positive and negative electric loadings on the energy release rate are shown in Fig. 6. These effects are mechanical-load-dependent.

7. Concluding Remarks

The energy release rate of an electrically insulated crack has been derived based on the energy equilibrium principle. We find that the energy release rate G does not equal to the J-integral. In general, G equals to the sum of the local J-integral around the crack tip and the average contribution from the electric field along the entire crack surface. Under purely electric loading, the energy release rate is always negative, indicating that the crack has a tendency to close. Under combined

mechanical and electrical loading, the energy release rate is positive when the applied mechanical load exceeds a critical value, which is a function of the strength of the applied electrical field. An electric field may decrease or increase the energy release rate depending on the strength of the electric field and the level of the mechanical load. The effect of electric field on the fracture behaviour of piezoelectric media is thus mechanical-load-dependent.

It should be pointed out that the present work is based on linear electro-elasticity. During the failure process of a piezoelectric material, the material, especially at the vicinity of a crack, behaves nonlinearly. Thus, the failure behavior of piezoelectric materials might deviate from the predictions based on linear approaches. Even so, the linear results are still foundational and important to the development of nonlinear analyses. Moreover, the linear results might be also used to predict the fracture behavior of piezoelectric materials at low levels of loading, at which the nonlinear behavior might occur at a very small crack-processing zone and thus, the materials could reveal a macroscopically linear fracture behavior.

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