Light amplification and localization in random amplifying layered media: Statistics from physical solutions

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(Received 17 December 2001; revised manuscript received 1 May 2002; published 1 August 2002)

Statistical properties of light propagation in finite-sized random amplifying layered media are studied by selecting the physical solutions according to a criterion in the framework of the time-independent Maxwell's equation. The criterion is demonstrated to be qualitatively correct. We find that the probability of the physical solution, P , decays exponentially with sample size L . Both P and the statistics of the reflection and the transmission coefficients obey a two-parameter scaling, i.e., $\Lambda \equiv L/\xi_o$ and $q \equiv \xi_o/l_g$, where ξ_o is the localization length in the absence of gain and l_g is the gain length. It is also found that, in the limit of large L , the localization length remains unchanged in the presence of gain.

DOI: 10.1103/PhysRevB.66.073101

PACS number(s): 42.25.Bs, 72.15.Rn, 78.45.+h

In the past few years, light propagation in random amplifying media has been the focus of extensive theoretical^{1–10} and experimental^{11–13} studies. This interest arose from the idea of random lasers, where the gain threshold can be substantially reduced due to the coherent feedback caused by recurrent multiple scattering in random media as well as the localization of waves. Experimentally, laserlike spectral narrowing due to amplified spontaneous emission was first observed in certain random media with gain.¹² Recently, laser action has been reported in the strong scattering media of semiconductor powders, where, in addition to spectral narrowing, discrete lasing modes have been observed due to recurrent scattering.¹³ Most theoretical studies on wave propagation in amplifying random media and properties of random laser were conducted based on the solutions obtained from a time-independent Maxwell's equation.^{3–5}

However, in a recent study using time-dependent Maxwell's equation, it has been found that the amplitude of both transmitted and reflected wave diverges when the system is above the lasing threshold.⁷ This implies that the results of transmission and reflection coefficient obtained from the time-independent Maxwell's equation are physical only when the system is below the threshold. However, in a finite-sized random amplifying layered medium with a fixed incident wave frequency, the threshold value is configuration dependent. Thus, we have two interesting questions. First, what is the probability of finding physical solutions among an ensemble of random configurations and how this probability varies with sample size, frequency of the incident wave, and characteristics of the random media. Second, what are the statistical distributions of the transmitted and reflected intensities arising from physical solutions. Since solving a time-dependent equation is very time consuming, it is practically impossible to answer these questions by using the time-dependent approach. A feasible approach, instead, is to find a reliable criterion that allows us to separate a physical solution from an unphysical one in the framework of time-independent equation.

In this work, such a criterion is proposed. In this criterion, a solution is considered as unphysical when it lies beyond the first maximum of the transmission peak as a function of the gain coefficient. Therefore, the lasing threshold is con-

sidered to be at the first peak of the transmission coefficient. The physical reason behind this criterion is the following. As the gain coefficient is increased, a resonant pole close to the incident frequency will move from the lower frequency plane to the upper frequency plane. The transmission becomes a peak when the pole reaches the real axis. Thus, we consider the solution unphysical when the gain coefficient lies beyond the first transmission peak as the poles has already moved to the upper frequency plane and the system become anticausal. The criterion is valid when the frequency is close to a resonant pole. However, when the frequency is away from a resonant pole, the interactions from other nearby poles may make the criterion less valid. For this reason, we have to test the validity of this criterion. We have done so in three different layered media by comparing the values of the lasing thresholds obtained from both time-dependent and time-independent equations at different frequency positions of a resonant pole. The results of the testing have demonstrated that the proposed criterion is qualitatively correct. Based on this criterion, we have reinvestigated the statistical properties of wave amplification and localization in random layered media by separating the physical solutions from the unphysical ones. This criterion also allows us to study the effect of gain on the localization length. Previous simulation results⁵ as well as the analytical study by Paasschens *et al.*⁴ indicated that the localization length would be reduced in the presence of gain. However, these results may be due to the presence of unphysical solutions, which are predominant at large sample thickness. Thus, a reinvestigation of this problem by using only the physical solution would provide a more convincing answer to this interesting and important question.

In order to establish the validity of the criterion described above, we first consider the case of a Fabry-Perot system. The one-dimensional time-independent Maxwell's equation can be written as

$$\frac{d^2 E(z)}{dz^2} + \frac{\omega^2}{c^2} \varepsilon(z) E(z) = 0, \quad (1)$$

where E is the electric field, ω is the wave frequency, $\varepsilon(z) = \varepsilon'(z) - i\varepsilon''(z)$ is the complex dielectric constant, and c is

FIG. 1. In a Fabry-Perot system, the percentage error of the lasing threshold introduced by the criterion proposed in this work. Inset, Probability density of the reflection coefficient for the case $q=0.5$ and $\Lambda=\Lambda_c \approx 1.58$ (open symbols) and $\Lambda=0.6 < \Lambda_c$ (solid symbols). The solid line represents $0.466 \exp(-1/x)/x^2$ with $x=(R-1)/2q$.

the speed of light in vacuum and is set as unity for convenience. Inside the Fabry-Perot system ($0 < z < L$), ϵ is a constant with $\epsilon' > 1$ and $\epsilon'' > 0$ for amplification. For the background, we take $\epsilon = 1$. The transmission amplitude of such a system takes the form $t = (1 - r^2)e^{ikL}/(1 - r^2e^{2ikL})$, where $r = (k - k_o)/(k + k_o)$, $k_o = \omega = 2\pi/\lambda$ and $k = k' - ik'' = k_o\sqrt{\epsilon' - i\epsilon''}$. When $\epsilon'' \ll \epsilon'$, we have $k' \approx k_o\sqrt{\epsilon'}$ and $k'' \approx k_o\epsilon''/2\sqrt{\epsilon'}$. From the time-dependent solution of the Fabry-Perot system, it is known that lasing occurs when $|r^2e^{2ikL}| = 1$,⁶ which implies a threshold gain of $\epsilon_c'' \approx -2\sqrt{\epsilon'} \ln|r|/(k_oL)$. However, by using our criterion, the threshold value, $\tilde{\epsilon}_c''$, should be obtained from the smallest ϵ'' that satisfies the condition $d^2|t|^2/d\epsilon'' = 0$. As an example, we choose $\epsilon' = 9$ and determine $\tilde{\epsilon}_c''$ as a function of $1/(k_oL)$. In Fig. 1, we plot the percentage error introduced by the criterion, denoted as $\Delta\epsilon_c''/\epsilon_c'' \equiv |\tilde{\epsilon}_c'' - \epsilon_c''|/\epsilon_c''$, where the circles and triangles denote, respectively, the frequencies at the peaks and valleys of various resonances. Very small error is found across the entire spectrum. The error reduces quadratically with increasing k_oL . In fact, at a resonant peak, it is easy to show that

$$\Delta\epsilon_c''/\epsilon_c'' \approx 4/(\epsilon' - 1)^2(1/k_oL)^2 \approx [1/(\ln|r|)^2(\epsilon' - 1)^2] \times (\epsilon''/\sqrt{\epsilon'})^2.$$

Thus, the criterion is very accurate when the critical gain is small, i.e., $\epsilon_c'' \ll \sqrt{\epsilon'}$.

In addition to the Fabry-Perot system, we have also tested the criterion in other layered systems, such as binary and random systems. In such systems, the value of ϵ_c'' is numerically determined by solving the time-dependent equation using a finite-difference-time-domain (FDTD) method, which is in the fourth order in both time and space.¹⁴ A semi-infinite plane wave of a fixed frequency is incident from one side

FIG. 2. Time-resolved transmitted intensity from a Fabry-Perot system in vacuum with a plane-wave incidence at various values of gain parameter, ϵ'' . Other parameters are $L=5$, $\epsilon'=9$, and $k_o=33.78$. t_o is the two-way travel time.

and the time-resolved intensity is measured at the other side. Our FDTD scheme has been successfully tested against the known result of the Fabry-Perot system. As an example, for an arbitrarily chosen set of parameters ($L=5$, $\epsilon'=9$, and $k_o=33.78$), the FDTD results are shown in Fig. 2 for various gain parameters ranging from $\epsilon''=0.023$ to 0.027 , where $t_o = 2\sqrt{\epsilon'}L = 30$ is the two-way travel time. Figure 2 predicts a threshold gain that falls between 0.024 and 0.025 . This is in excellent agreement with the exact result of $\epsilon_c'' \approx 0.0246$. In order to determine the value of $\tilde{\epsilon}_c''$ for binary and random systems, we calculate the transmission coefficient, $T \equiv |t|^2$, as a function of ϵ'' by using the transfer-matrix method.⁵ The location of the first maximum of T determines the value of $\tilde{\epsilon}_c''$. In the case of binary layers, we consider a system of $L=100$ alternating layers of A and B types, with each layer having a thickness of $a=1$ and dielectric constants of $\epsilon_A=2$ and $\epsilon_B=1$. In the case of random layers, we choose $L=100$ and 150 . The dielectric constant in each layer is a random number uniformly distributed in the interval $[0.7, 2.3]$. From the transmission spectra of the above systems, we have randomly chosen six resonant peaks. For each resonant peak, we pick three frequencies: one near the peak. One near the middle, and one near the valley. At each frequency we determine both ϵ_c'' and $\tilde{\epsilon}_c''$ by using the FDTD and the transfer-matrix methods, respectively. The above results are shown in Fig. 3. All the data points show that the values of $\tilde{\epsilon}_c''$ have the same magnitude as the values of ϵ_c'' , indicating that the criterion is qualitatively correct. Thus, we can use this criterion to study how the probability of physical solutions decays as a function of L at a fixed value of ϵ'' . When L approaches infinity, it is expected that ϵ_c'' decrease to zero and, that, therefore, the probability approaches zero as well.

The random layered media we consider are the following.⁵ For a sample of L layers, the thickness Δz of each layer is a random number that obeys a distribution $\exp(-\Delta z)$, with the mean thickness of unity. In each layer,

FIG. 3. Lasing threshold calculated by the time-dependent (ε_c) and the time-independent ($\bar{\varepsilon}_c$) equations for both a binary system and a random system of different size L . Calculations are performed at three frequencies near a resonant peak for each of the six randomly chosen resonances.

the real part of the dielectric constant has the value $\varepsilon_n = 1 + \sigma \eta_n$, where $0 < \sigma < 1$ gives the magnitude of randomness, and η_n is a random number between $[-1, 1]$. The imaginary part is assumed to have a constant value, $-\varepsilon''$, across the sample. The entire sample is embedded in a homogeneous background with dielectric constant unity. We focus our study on the following three cases: (a) wavelength $\lambda = 10$ and randomness $\sigma = 0.3$; (b) $\lambda = 25$ and $\sigma = 0.9$; and (c) $\lambda = 50$ and $\sigma = 0.9$. In the absence of gain, the corresponding localization lengths, ξ_o , for the three cases are 861, 281, and 987, respectively. The localization length, which is defined as $L \rightarrow \infty$ limit of $\xi(L) = -2L/\ln T(L)$, can be obtained from the configurational average of $\langle 1/\xi(L) \rangle^{-1}$.⁵ In the presence of gain, we can define a dimensionless gain length as $q^{-1} \equiv l_g/\xi_o$ with the gain length $l_g = 1/k'' \approx \lambda/\pi\varepsilon''$.⁵ In our simulations, for each $\xi_o(\lambda, \sigma)$, we have chosen five gain parameters, q , such that q has the values 0, 1, 0.5, 1, 3, and 10. For each set of parameters, $\{\xi_o, q\}$, we calculate the transmission coefficient, T , for a sample of size L in any given random configuration. In order to determine whether this solution is physical or unphysical, we gradually reduce the gain parameter and recalculate the transmission coefficient. According to the criterion, if the transmission coefficient decreases monotonically with decreasing ε'' , the solution of T is considered to be physical. Otherwise, it is unphysical. The process is repeated for 20 000 to 40 000 configurations at each size L . At $q = 0.5$, the probability of physical solutions, $P_{q=0.5}(L/\xi_o)$, as a function of dimensionless size, $\Lambda \equiv L/\xi_o$, is shown in Fig. 4, with open squares, open triangles, and open circles for cases (a), (b), and (c), respectively. The same symbols will be used in the later figures. It is interesting to see that all three sets of data points fall on a single negative exponential curve (solid line), indicating an exponential decay in the probability of physical solutions with a single decay length, Λ_o . The solid symbols denote the probability of unphysical solutions, i.e., $1 - P$. In the inset of Fig. 4, we replot $P_{q=0.5}(L/\xi_o)$ in semi-log scale. From the

FIG. 4. Probability of physical solutions, P , as a function of dimensionless sample size, L/ξ_o , for the cases of (a) $\lambda = 10$ and $\sigma = 0.3$ (open squares), (b) $\lambda = 25$ and $\sigma = 0.9$ (open triangles), and (c) $\lambda = 50$ and $\sigma = 0.9$ (open circles). The corresponding solid symbols denote the probability of unphysical solutions. The inset plots the logarithm of P . The dotted line represents a fitted curve.

slope of the fitted dotted line we can obtain Λ_o . The above calculations are repeated for other values of q and similar results have been found, except that the decay length, Λ_o , varies with q . In Fig. 5, we plot the function, $\Lambda_o(q)$, in solid circles using log-log scales. The data points seem to follow a straight line with a slope of $-2/3$ as shown by the dotted line.

We have also studied the probability density of reflection/transmission coefficient at various values of q and Λ for all three cases. In the inset of Fig. 1, we plot the probability density of reflection coefficient $P(x)$ as a function of a scaled variable, $x = (R - 1)/2q$, for the parameters $q = 0.5$

FIG. 5. Dimensionless decay length, Λ_o , of the probability of physical solutions is plotted as a function of dimensionless gain length, $q^{-1} \equiv l_g/\xi_o$, as solid circles in a log-log scale. Open symbols denote the renormalized critical size, $\Lambda_c \equiv L_c/\xi_o$, at different values of q . The dotted line has a slope of $-2/3$.

FIG. 6. Localization length in the presence of gain, ξ , for different sample sizes. Open symbols are results obtained from physical solutions. Solid symbols are obtained when the unphysical solutions are included.

and $\Lambda = 0.60$ and 1.58 with solid and open symbols, respectively. There are two points we would like to make here. First, we find that all three sets of data points fall on a single solid curve. This implies that the two-parameter (Λ and q) scaling theory found previously in this problem remains valid for physical solutions.^{3,5} Such a two-parameter scaling is also indicated in Fig. 4. Second, the solid curve in the inset of Fig. 1 for the case of $\Lambda = 1.58$ (where the probability of physical solution, $P_{0.5}$, is 0.5) is proportional to the analytical results of $P_{\infty}(x) = \exp(-1/x)/x^2$ obtained previously for an infinite sample as shown by a solid curve.³ This implies that the averaged reflection coefficient, $\langle R \rangle$, diverges for physical solutions. This is also true for other values of q . For

the case of $q = 0.5$ and $\Lambda = 0.6$, $P(x)$ decays exponentially at large R 's leading to a finite value of $\langle R \rangle$. If we denote Λ_C as the value of Λ when $P_q(\Lambda_C) = 0.5$, the function $\Lambda_c(q)$ is plotted in Fig. 5 with open symbols. A straight line with a slope $-2/3$ implies the relation $\Lambda_c \propto q^{-2/3}$, which is different from the relation $\Lambda_c \propto q^{-1/2}$ found earlier when the unphysical solutions were also included.⁵ The $q^{-2/3}$ behavior found in Λ_C is consistent with the behavior found in Λ_o (solid circles). They both imply the same behavior between Λ and q when the probability of unphysical solution reaches a significant level. Since, in random media, the critical size, L_C , for laser action can only be defined in statistical terms. The $q^{-2/3}$ behavior found here seems to indicate a relation $L_c^3 \approx \xi_o l_g^2$, or $L_c^3 \approx l_g^2$. In the last relation, we have used the fact that, in one dimension, ξ_o is about a few times of the mean free path, l .⁵ The lasing condition obtained here is different from the well-known relation $L_c^2 \approx l_g$ obtained previously in both the diffusive and the localized regimes.^{1,5}

Finally, we have calculated the localization length in the presence of gain, $\xi(L)$, using only the physical solutions. The result is shown in Fig. 6 with open symbols. The trend $\xi(L)/\xi_o \rightarrow 1$ when $L \rightarrow \infty$ suggests that the localization length remains unchanged in the presence of gain. When the unphysical solutions are also included, the previous results show a reduction in the localization length in the presence of gain as shown with the solid symbols in Fig. 6.⁵

Our results indicate that the statistical properties obtained from physical solutions can be qualitatively different from those obtained when unphysical solutions are also included. The criterion proposed here suggests a method to select physical solutions for other one-dimensional model systems as well.³⁻⁵

The work is supported by Hong Kong RGC Grant No. HKUST 6158/01P.

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