Portfolio Loss Distribution
Risky assets in loan portfolio

- highly illiquid assets
- “hold-to-maturity” in the bank’s balance sheet

**Outstandings**
The portion of the bank asset that has already been extended to borrowers.

**Commitment**
A commitment is an amount the bank has committed to lend. Should the borrower encounter financial difficulties, it would draw on this committed line of credit.
**Adjusted exposure and expected loss**

Let $\alpha$ be the amount of drawn down or usage given default.

\[
\text{Asset value at later time } H, V_H = \text{Outstanding} + \alpha \times \text{commitment, Risky}
\]

\[
(1-\alpha) \times \text{commitment, Riskless}
\]

Adjusted exposure is the risky part of $V_H$.

Expected loss = adjusted exposure $\times$ loss given default $\times$ probability of default

* Normally, practitioners treat the uncertain draw-down rate as a known function of the obligor’s end-of-horizon credit class rating.
# Example calculation of expected loss

<table>
<thead>
<tr>
<th>Commitment</th>
<th>$10,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outstanding</td>
<td>$3,000,000</td>
</tr>
<tr>
<td>Internal risk rating</td>
<td>3</td>
</tr>
<tr>
<td>Maturity</td>
<td>1 year</td>
</tr>
<tr>
<td>Type</td>
<td>Non-secured</td>
</tr>
<tr>
<td>Unused drawn-down on default (for internal rating = 3)</td>
<td>65%</td>
</tr>
<tr>
<td>Adjusted exposure on default</td>
<td>$8,250,000</td>
</tr>
<tr>
<td>EDF for internal rating = 3</td>
<td>0.15%</td>
</tr>
<tr>
<td>Loss given default for non-secured asset</td>
<td>50%</td>
</tr>
<tr>
<td>Expected loss</td>
<td>$6,188</td>
</tr>
</tbody>
</table>
Unexpected loss

Unexpected loss is the estimated volatility of the potential loss in value of the asset around its expected loss.

\[
UL = AE \times \sqrt{EDF \times \sigma_{LGD}^2 + LGD^2 \times \sigma_{EDF}^2}
\]

where

\[
\sigma_{EDF}^2 = EDF \times (1 - EDF).
\]

Assumptions

* The random risk factors contributing to an obligor’s default (resulting in EDF) are statistically independent of the severity of loss (as given by LGD).

* The default process is two-state event.
Example on unexpected loss calculation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjusted exposure</td>
<td>$8,250,000</td>
</tr>
<tr>
<td>EDF</td>
<td>0.15%</td>
</tr>
<tr>
<td>$\sigma_{EDF}$</td>
<td>3.87%</td>
</tr>
<tr>
<td>LGD</td>
<td>50%</td>
</tr>
<tr>
<td>$\sigma_{LGD}$</td>
<td>25%</td>
</tr>
<tr>
<td>Unexpected loss</td>
<td>$178,511</td>
</tr>
</tbody>
</table>

* The calculated unexpected loss is 2.16% of the adjusted exposure, while the expected loss is only 0.075%.
Comparison between expected loss and unexpected loss

* The higher the recovery rate (lower LGD), the lower is the percentage loss for both EL and UL.

* EL increases linearly with decreasing credit quality (with increasing EDF)

* UL increases much faster than EL with increasing EDF.

![Graph showing comparison between EL and UL percentage loss per unit of adjusted loss.](image)
Assets with varying terms of maturity

* The longer the term to maturity, the greater the variation in asset value due to changes in credit quality.

* The two-state default process paradigm inherently ignores the credit losses associated with defaults that occur beyond the analysis horizon.

* To mitigate some of the maturity effect, banks commonly adjust a risky asset’s internal credit class rating in accordance with its terms to maturity.
Portfolio expected loss

\[ EL_p = \sum_i EL_i = \sum_i AE_i \times LGD_i \times EDF_i \]

where \( EL_p \) is the expected loss for the portfolio,
\( AE_i \) is the risky portion of the terminal value of the \( i \)th asset
to which the bank is exposed in the event of default.

We may write

\[ \frac{EL_p}{AE_p} = \sum_i w_i \frac{EL_i}{AE_i} \]

where the weights refer to

\[ w_i = \frac{AE_i}{\sum_i AE_i} = \frac{AE_i}{AE_p}. \]
\[
\frac{EL_p}{AE_p} = \sum \frac{EL_i}{AE_i} = \sum \frac{AE_i}{AE_p} \frac{EL_i}{AE_i} = w_i \frac{EL_i}{AE_i}
\]

<table>
<thead>
<tr>
<th>(i)</th>
<th>(AE_i)</th>
<th>(w_i)</th>
<th>(EL_i)</th>
<th>(EL_i/\ AE_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10\ M$</td>
<td>0.5</td>
<td>$1$</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>$4\ M$</td>
<td>0.2</td>
<td>$0.5$</td>
<td>0.125</td>
</tr>
<tr>
<td>3</td>
<td>$6\ M$</td>
<td>0.3</td>
<td>$0.6$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

\[\sum AE_i = $20\ M \quad \sum w_i = 1\]

\[
\frac{EL_p}{AE_p} = 0.5 \times 0.1 + 0.2 \times 0.125 + 0.3 \times 0.1 = 0.105
\]
Portfolio unexpected loss

portfolio unexpected loss = \( UL_p = \sqrt{\sum_i \sum_j \rho_{ij} w_i w_j UL_i UL_j} \)

where

\[ UL_i = AE_i \times \sqrt{EDF_i \times \sigma^2_{LGD_i} + LGD_i^2 \times \sigma^2_{EDF_i}} \]

and \( \rho_{ij} \) is the correlation of default between asset \( i \) and asset \( j \). Due to diversification effect, we expect

\[ UL_p \ll \sum_i UL_i. \]
Risk contribution

The risk contribution of a risky asset $i$ to the portfolio unexpected loss is defined to be the *incremental risk* that the exposure of a single asset contributes to the portfolio’s total risk.

$$\text{RC}_i = \text{UL}_i \frac{\partial \text{UL}_p}{\partial \text{UL}_i}$$

and it can be shown that

$$\text{RC}_i = \frac{\text{UL}_i \sum_j \text{UL}_j \rho_{ij}}{\text{UL}_p}.$$
Undiversifiable risk

The risk contribution is a measure of the *undiversifiable risk* of an asset in the portfolio – the amount of credit risk which cannot be diversified away by placing the asset in the portfolio.

\[ UL_p = \sum_i RC_i \]

To incorporate industry correlation, using \( i \rightarrow \text{industry } \alpha \) and \( j \rightarrow \text{industry } \beta \)

\[ RC_i = \frac{UL_i}{UL_p} \left[ UL_{i\in\alpha} (1 - \rho_{\alpha\alpha}) + \sum_{\beta \neq \alpha} \left( \sum_{k \in \beta} UL_k \right) \rho_{\alpha\beta} \right]. \]
Calculation of EL, UL and RC for a two-asset portfolio

\[ \rho \] default correlation between the two exposures

\[ \text{EL}_p \] portfolio expected loss

\[ \text{EL}_p = \text{EL}_1 + \text{EL}_2 \]

\[ \text{UL}_p \] portfolio unexpected loss

\[ \text{UL}_p = \sqrt{\text{UL}_1^2 + \text{UL}_2^2 + 2 \rho \text{UL}_1 \text{UL}_2} \]

\[ \text{RC}_1 \] risk contribution from Exposure 1

\[ \text{RC}_1 = \text{UL}_1 (\text{UL}_1 + \rho \text{UL}_2) / \text{UL}_p \]

\[ \text{RC}_2 \] risk contribution from Exposure 2

\[ \text{RC}_2 = \text{UL}_2 (\text{UL}_2 + \rho \text{UL}_1) / \text{UL}_p \]

\[ \text{UL}_p = \text{RC}_1 + \text{RC}_2 \]

\[ \text{UL}_p << \text{UL}_1 + \text{UL}_2 \]
Fitting of loss distribution

The two statistical measures about the credit portfolio are

- portfolio expected loss;
- portfolio unexpected loss.

At the simplest level, the *beta distribution* may be chosen to fit the portfolio loss distribution.

*Reservation* A beta distribution with only two degrees of freedom is perhaps insufficient to give an adequate description of the tail events in the loss distribution.
Beta distribution

The density function of a beta distribution is

\[
F(x, \alpha, \beta) = \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}, \quad \alpha > 0, \beta > 0
\]

Mean \( \mu = \frac{\alpha}{\alpha + \beta} \) and variance \( \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \).
Economic Capital

If $X_T$ is the random variable for loss and $z$ is the percentage probability (confidence level), what is the quantity $v$ of minimum economic capital EC needed to protect the bank from insolvency at the time horizon $T$ such that

$$\Pr[X_T \leq v] = z.$$ 

Here, $z$ is the desired debt rating of the bank, say, 99.97% for an AA rating.
frequency of loss

\[ X_T \]

\[ \text{UL}_p \]

\[ \text{EL}_p \]

\[ \text{EC} \]
Capital multiplier

Given a desired level of $z$, what is $EC$ such that

$$\Pr[X_T - EL_\rho \leq EC] = z.$$ 

Let $CM$ (capital multiplier) be defined by

$$EC = CM \times UL_\rho$$

then

$$\Pr\left[\frac{X_T - EL_\rho}{UL_\rho} \leq CM\right] = z.$$
Monte Carlo simulation of loss distribution of a portfolio

1. **Estimate default and losses**
   - Assign risk ratings to loss facilities and determine their default probability
   - Assign LGD and $\sigma_{LGD}$

2. **Estimate asset correlation between obligors**
   - Determine pairwise asset correlation whenever possible
   - OR
   - Assign obligors to industry groupings, then determine industry pair correlation
3. Generate random loss given default

Determine stochastic loss given default

4. Generate correlated default events

Correlated default events + Decomposition of covariance matrix + Simulate default point
5. **Loss calculation**

Calculate facility loss for each scenario and obtain portfolio loss

6. **Loss distribution**

Construct simulated portfolio loss distribution
Generation of correlated default events

- Generate a set of random numbers drawn from a standard normal distribution.

- Perform a decomposition (Cholesky, SVD or eigenvalue) on the asset correlation matrix to transform the independent set of random numbers (stored in the vector $\mathbf{e}$) into a set of correlated asset values (stored in the vector $\mathbf{e}'$). Here, the transformation matrix is $\mathbf{M}$, where

$$\mathbf{e}' = \mathbf{M} \mathbf{e}.$$ 

The covariance matrix $\Sigma$ and $\mathbf{M}$ are related by

$$\mathbf{M}^T \mathbf{M} = \Sigma.$$
Calculation of the default point

The default point threshold, $DP$, of the $i^{th}$ obligor can be defined as $DP = \mathcal{N}^{-1}(\text{EDF}_i, 0, 1)$. The criterion of default for the $i^{th}$ obligor is

$$\begin{align*}
\text{default} & \quad \text{if} \quad e'_i < DP_i \\
\text{no default} & \quad \text{if} \quad e'_i \geq DP_i.
\end{align*}$$
Generate loss given default

The LGD is a stochastic variable with an unknown distribution.

A typical example may be

<table>
<thead>
<tr>
<th>Recovery rate (%)</th>
<th>LGD (%)</th>
<th>$\sigma_{\text{LGD}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>secured</td>
<td>65</td>
<td>35</td>
</tr>
<tr>
<td>unsecured</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

$$\text{LGD}_i = \text{LGD}_s + f_i \times \sigma_{\text{LGD}}^s$$

where $f_i$ is drawn from a uniform distribution whose range is selected so that the resulting LGD has a standard deviation that is consistent with historical observation.
Calculation of loss

Summing all the simulated losses from one single scenario

\[
\text{Loss} = \sum_{\text{Obligors in default}} \text{Adjusted exposure}_i \times \text{LGD}
\]

Simulated loss distribution

The simulated loss distribution is obtained by repeating the above process sufficiently number of times.
Features of portfolio risk

- The variability of default risk within a portfolio is substantial.
- The correlation between default risks is generally low.
- The default risk itself is dynamic and subject to large fluctuations.
- Default risks can be effectively managed through diversification.
- Within a well-diversified portfolio, the loss behavior is characterized by lower than expected default credit losses for much of the time, but very large losses which are incurred infrequently.