Chern-Simons invariant in the Berry phase of a 2×2 Hamiltonian

Wu-Yi Hsiang

Department of Mathematics, Hong-Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Dung-Hai Lee

Department of Physics, University of California, Berkeley, California 94720

and Center for Advanced Study, Tsinghua University, Beijing, China

(Received 18 April 2001; published 1 October 2001)

By varying (x, y, z) within a manifold $\mathcal{M}$, the positive (negative)-energy eigenvectors of the 2×2 Hamiltonian $H = x\sigma_x + y\sigma_y + z\sigma_z$ (where $\sigma_{x,y,z}$ are the Pauli matrices) form a $U(1)$ fiber bundle. For certain $\mathcal{M}$ the bundle has nontrivial topology. For example, when $\mathcal{M} = S^2$ the associated bundle has nonzero Chern number indicating that it is topologically nontrivial at the highest level. In this paper we construct a simple 2×2 Hamiltonian whose eigenvector bundle exhibits a more subtle topological nontriviality when $\mathcal{M}$ is a closed three manifold. This nontrivial topology is characterized by nonzero Chern-Simons invariant.

DOI: 10.1103/PhysRevA.64.052101

PACS number(s): 03.65.Vf, 05.45.-a

Since its discovery in 1984 [1], the Berry phase has played an important role in quantum mechanics. For a simple example of the Berry phase, consider the following 2×2 Hamiltonian:

$$H(\mathbf{r}) = x\sigma_x + y\sigma_y + z\sigma_z \equiv \mathbf{r} \cdot \boldsymbol{\sigma}, \quad (1)$$

where $\sigma_{x,y,z}$ are the three components of the Pauli matrices and x, y, z are real parameters. For a fixed $\mathbf{r} = (x, y, z)$ the Hamiltonian in Eq. (1) has two eigenvalues $E_{\pm} = \pm \sqrt{x^2 + y^2 + z^2} \equiv \pm |\mathbf{r}|$.

Let us focus on the eigenvector $|\psi(\mathbf{r})\rangle$ associated with the positive eigenvalue for the rest of the paper. The Berry phase induced by an adiabatic evolution of $\mathbf{r}$ around a closed loop $\mathcal{C}$ is given by

$$\gamma = \oint_{\mathcal{C}} dx^\mu A_\mu^b(\mathbf{r}), \quad (2)$$

where

$$A_\mu^b(\mathbf{r}) = \frac{1}{i} \langle \psi(\mathbf{r}) | \partial_\mu \psi(\mathbf{r}) \rangle. \quad (3)$$

It turns out that the A_μ^b in Eq. (3) has a geometric interpretation as we shall explain in what follows.

At a fixed $\mathbf{r}$ if one is given a spinor (i.e., a two-component column vector) satisfying

$$\begin{aligned} H(\mathbf{r})|\psi(\mathbf{r})\rangle &= |\mathbf{r}||\psi(\mathbf{r})\rangle, \\ \langle \psi(\mathbf{r}) | \psi(\mathbf{r}) \rangle &= 1, \end{aligned} \quad (4)$$

one can generate a continuum of other spinors that satisfy Eq. (4) by the transformation

$$|\psi(\mathbf{r})\rangle \rightarrow e^{i\theta} |\psi(\mathbf{r})\rangle. \quad (5)$$

This family of $|\psi(\mathbf{r})\rangle$ spans an internal space that is invariant under a $U(1)$ group of transformations [Eq. (5)]. As $\mathbf{r}$ varies through a manifold $\mathcal{M}$ (henceforth referred to as the base

space) the internal space sweeps out a geometric object called “fiber bundle.” Since $U(1)$ leaves the internal space invariant this fiber bundle is a $U(1)$ bundle. In what follows we shall refer to such fiber bundle as the eigenbundle of Eq. (1).

It turns out that Eq. (3) is precisely the geometric connection on the eigenbundle [2]. The connection A_μ^b defined above is not unique. Indeed, by performing the transformation $|\psi(\mathbf{r})\rangle \rightarrow e^{i\theta(\mathbf{r})} |\psi(\mathbf{r})\rangle$ we induce a “gauge transformation” on A_μ^b ,

$$A_\mu^b \rightarrow A_\mu^b + \partial_\mu \theta. \quad (6)$$

It is obvious that the Berry phase [Eq. (2)] is gauge invariant.

Next we shall focus on two-dimensional base spaces $\mathcal{M}$ that are closed surfaces. It turns out that if $\mathcal{M}$ encloses the origin (for example, $\mathcal{M} = S^2 = \{\mathbf{r}; |\mathbf{r}| = 1\}$), it is impossible to choose a gauge in which A_μ^b is nonsingular everywhere. In order to obtain locally nonsingular A_μ^b it is necessary to divide $\mathcal{M}$ into a number of (overlapping) patches so that (1) A_μ^b is nonsingular in each patch, and (2) in the region where two patches overlap the different A_μ^b 's differ only by a gauge transformation. Historically this problem was encountered by Dirac when he tried to write down the vector potential in the neighborhood of a magnetic monopole [3]. It turns out that under the framework of quantum mechanics, condition (2) requires the strength of the monopole to be quantized [3,4].

In geometry it is known that the nonexistence of an everywhere-nonsingular connection is the manifestation of nontrivial topology. In his seminal work S.S. Chern discovered a set of invariants to characterize such nontriviality [5]. For the simple case we are considering the invariants reduce to a single number, the Chern number

$$C = \frac{1}{4\pi} \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} F_{\mu\nu}^b. \quad (7)$$

Here $F_{\mu\nu}^b = \partial_\mu A_\nu^b - \partial_\nu A_\mu^b$ is the curvature associated with A_μ^b . For the eigenbundle of Eq. (1) it is simple to show that $C = 1/2$ or 0 depending on whether $\mathcal{M}$ encloses the origin. If we interpret $F_{\mu\nu}^b$ as magnetic field, the above result suggests

that $\mathcal{C}$ is the total magnetic flux (through $\mathcal{M}$) produced by a magnetic monopole located at $\mathbf{r}=\mathbf{0}$. In a proof similar but more general than that given in Ref. [3], Chern showed that $\mathcal{C}$ should be quantized to values $n/2$ where n is an integer [5].

Since $\mathcal{C} > 1/2$ is allowed, it is interesting to ask what kind of Hamiltonian will have $\mathcal{C}=n/2$ ($n > 1$) eigenbundles. One answer is given by the following $(n+1) \times (n+1)$ matrix:

$$H(\mathbf{r}) = \mathbf{r} \cdot \mathbf{S}, \quad (8)$$

where $\mathbf{S}=(S_x, S_y, S_z)$ are the matrices representing the three generators of $SU(2)$ in the spin $S=n/2$ representation. For example, for $n=2$ we have

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (9)$$

There is another way of modifying Eq. (1) that also leads to $\mathcal{C}=n/2$ eigenbundle. Interestingly this time we do not need to enlarge the dimension of H . Consider the following 2×2 matrix:

$$H(\mathbf{r}) = \hat{\mathbf{h}}(\mathbf{r}) \cdot \boldsymbol{\sigma}, \quad (10)$$

where $\hat{\mathbf{h}}(\mathbf{r})$ is a suitably chosen unit vector field that defines a mapping from $\mathcal{M}$ (a closed two manifold) to S^2 . It is known that such mappings can be classified into homotopy classes, each labeled by an integer

$$\mathcal{P} = \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} J_{\mu\nu}. \quad (11)$$

Here the Pontryagin form $J_{\mu\nu}$ is given by

$$J_{\mu\nu} = \frac{1}{4\pi} \hat{\mathbf{h}} \cdot (\partial_\mu \hat{\mathbf{h}} \times \partial_\nu \hat{\mathbf{h}}). \quad (12)$$

We will later show that by choosing a $\hat{\mathbf{h}}(\mathbf{r})$ with $\mathcal{P}=n$ the eigenbundle of Eq. (10) exhibits $\mathcal{C}=n/2$.

The Chern number records the highest level of topological nontriviality. When the Chern number vanishes the eigenbundle can still be nontrivial at a more subtle level. Let us consider a closed three manifold in which $\mathcal{C}=0$ for all closed sub two manifolds, which implies the absence of monopole. Without monopole the “flux lines” associated with the vector field $\epsilon^{\mu\nu\lambda} F_{\nu\lambda}$ form closed loops. Under this condition there is a topological interesting situation in which these flux lines link with one another. It is clear that this class of eigenbundles are topologically distinct from those without linking flux lines.

In 1974 Chern and Simons discovered an invariant, the Chern-Simons invariant, that quantifies this more subtle level

of topological nontriviality [6]. For a closed three-manifold $\mathcal{M}$ the Chern-Simons invariant is $\mathcal{I}_{CS}$ given by

$$\mathcal{I}_{CS} = \frac{1}{4\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A_\mu^b \partial_\nu A_\lambda^b. \quad (13)$$

We note that when $\mathcal{M}$ is closed $\mathcal{I}_{CS}$ is gauge invariant. The topological information recorded by $\mathcal{I}_{CS}$ is precisely the linking between the flux lines. The fact that linking is only defined in three dimensions explains why the Chern-Simons invariant requires three-dimensional base space.

Since there is another level of topological nontriviality, it is natural to ask whether one can modify Eq. (1) so that the eigen bundle exhibits such topology. We shall prove that the Hamiltonian given by Eq. (10) also works so long as $\hat{\mathbf{h}}(\mathbf{r})$ is chosen appropriately.

Now let us restrict ourselves to the case where the base space $\mathcal{M}$ is a simply connected closed three manifold and $\mathbf{r}$ labels the points in it. In such case $\hat{\mathbf{h}}(\mathbf{r})$ is a mapping from a simply connected closed three manifold to S^2 . The work of Hopf shows that such mapping can also be classified into homotopy classes by an integral valued Hopf invariant $\mathcal{H}$. A Hopf map is a smooth configuration of $\hat{\mathbf{h}}(\mathbf{r})$. Due to the fact that there are only two linearly independent directions for $\partial_\mu \hat{\mathbf{h}}$, it follows

$$\epsilon^{\mu\nu\lambda} \partial_\mu J_{\nu\lambda} = 0. \quad (14)$$

As the result the flow lines associated with $\epsilon^{\mu\nu\lambda} J_{\nu\lambda}$ form closed loops. For a nontrivial Hopf map any pair of J loops link with each other. Because of Eq. (14) there exists a A_μ^h so that

$$J_{\mu\nu} = \frac{1}{4\pi} (\partial_\mu A_\nu^h - \partial_\nu A_\mu^h). \quad (15)$$

The Hopf invariant is simply the Chern-Simons invariant for A_μ^h , [7,8], i.e.,

$$\mathcal{H} = \frac{1}{4\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A_\mu^h \partial_\nu A_\lambda^h. \quad (16)$$

In the rest of the paper we prove the following.

(i) For closed two-manifold $\mathcal{M}$ the eigenbundle of

Eq. (10) has $\mathcal{C}=n/2$ if $\hat{\mathbf{h}}(\mathbf{r})$ has $\mathcal{P}=n$.

(ii) For closed three-manifold $\mathcal{M}$ the eigenbundle of

Eq. (10) has $\mathcal{I}_{CS}=n$ if $\hat{\mathbf{h}}(\mathbf{r})$ has $\mathcal{H}=n$. (17)

The proof amounts to show the following identity:

$$F_{\mu\nu}^b = 4\pi J_{\mu\nu}. \quad (18)$$

The Berry curvature is given by

$$\begin{aligned}
F_{\mu\nu}^b &= \frac{1}{i} [\partial_\mu \langle \psi_+ | \partial_\nu \psi_+ \rangle - \partial_\nu \langle \psi_+ | \partial_\mu \psi_+ \rangle] \\
&= \frac{1}{i} [\langle \partial_\mu \psi_+ | \partial_\nu \psi_+ \rangle - \langle \partial_\nu \psi_+ | \partial_\mu \psi_+ \rangle]. \quad (19)
\end{aligned}$$

To compute $\langle \partial_\mu \psi_+ | \partial_\nu \psi_+ \rangle - \langle \partial_\nu \psi_+ | \partial_\mu \psi_+ \rangle$ we insert a complete set of states ($I = \sum_{n=\pm} |\psi_n\rangle \langle \psi_n|$), and that gives

$$\begin{aligned}
&\langle \partial_\mu \psi_+ | \partial_\nu \psi_+ \rangle - \langle \partial_\nu \psi_+ | \partial_\mu \psi_+ \rangle \\
&= \sum_{n=\pm} \langle \partial_\mu \psi_+ | \psi_n \rangle \langle \psi_n | \partial_\nu \psi_+ \rangle - [\mu \leftrightarrow \nu] = \langle \partial_\mu \psi_+ | \psi_- \rangle \\
&\quad \times \langle \psi_- | \partial_\nu \psi_+ \rangle - [\mu \leftrightarrow \nu]. \quad (20)
\end{aligned}$$

In reaching the last line we have used the fact that $\langle \partial_\mu \psi_+ | \psi_+ \rangle \langle \psi_+ | \partial_\nu \psi_+ \rangle - [\mu \leftrightarrow \nu] = 0$.

To compute $\langle \psi_- | \partial_\nu \psi_+ \rangle$ in Eq. (20) we express the eigenvector of $H(\mathbf{r}') = H(\mathbf{r} + \delta\mathbf{r}) = H(\mathbf{r}) + \delta x^\lambda \partial_\lambda H$ in terms of those of $H(\mathbf{r})$ via first-order perturbation theory. Up to the first order in δx^μ , we obtain

$$\begin{aligned}
|\psi_+(\mathbf{r}')\rangle &= \left[|\psi_+\rangle + \frac{\langle \psi_- | \delta x^\lambda \partial_\lambda H | \psi_+ \rangle}{E_+ - E_-} |\psi_-\rangle \right] \\
&= \left[|\psi_+\rangle + \frac{\langle \psi_- | \delta x^\lambda \partial_\lambda \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle}{2} |\psi_-\rangle \right]. \quad (21)
\end{aligned}$$

Equation (21) implies that

$$\langle \psi_- | \partial_\nu \psi_+ \rangle = \frac{\langle \psi_- | \partial_\nu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle}{2}. \quad (22)$$

As the result we have

$$\begin{aligned}
&\langle \partial_\mu \psi_+ | \psi_- \rangle \langle \psi_- | \partial_\nu \psi_+ \rangle - [\mu \leftrightarrow \nu] \\
&= \frac{1}{4} \left[\langle \psi_+ | \partial_\mu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_- \rangle \langle \psi_- | \partial_\nu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle - [\mu \leftrightarrow \nu] \right] \\
&= \frac{1}{4} \sum_{n=\pm} \left[\langle \psi_+ | \partial_\mu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_n \rangle \langle \psi_n | \partial_\nu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle - [\mu \leftrightarrow \nu] \right] \\
&= \frac{1}{4} (\langle \psi_+ | [\partial_\mu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma}, \partial_\nu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma}] | \psi_+ \rangle) \\
&= \frac{i}{2} \epsilon_{abc} (\partial_\mu \hat{\mathbf{h}}_a) (\partial_\nu \hat{\mathbf{h}}_b) [\langle \psi_+ | \sigma_c | \psi_+ \rangle] \\
&= \frac{i}{2} \epsilon_{abc} (\partial_\mu \hat{\mathbf{h}}_a) (\partial_\nu \hat{\mathbf{h}}_b) \hat{\mathbf{h}}_c \\
&= \frac{i}{2} \hat{\mathbf{h}} \cdot (\partial_\mu \hat{\mathbf{h}} \times \partial_\nu \hat{\mathbf{h}}). \quad (23)
\end{aligned}$$

Going from the second to the third line of Eq. (23) we have used the fact that $\langle \psi_+ | \partial_\mu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle \langle \psi_+ | \partial_\nu \hat{\mathbf{h}} \cdot \boldsymbol{\sigma} | \psi_+ \rangle - [\mu \leftrightarrow \nu] = 0$. Substituting Eq. (20) and Eq. (23) into Eq. (19) we obtain

$$F_{\mu\nu}^b = \frac{1}{2} \hat{\mathbf{h}} \cdot \partial_\mu \hat{\mathbf{h}} \times \partial_\nu \hat{\mathbf{h}} = 4\pi J_{\mu\nu}. \quad (24)$$

After establishing Eq. (18) it is simple to prove (i) and (ii) of Eq. (17). For (i) the Chern number is given by

$$\mathcal{C} = \frac{1}{4\pi} \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} F_{\mu\nu}^b = \frac{1}{2} \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} J_{\mu\nu} = \mathcal{P}/2. \quad (25)$$

Thus $\mathcal{P} = n$ implies $\mathcal{C} = n/2$. Now let us prove (ii) of Eq. (17). Eq. (15) and Eq. (18) imply that

$$F_{\mu\nu}^b = 4\pi J_{\mu\nu} = F_{\mu\nu}^h. \quad (26)$$

Thus A_μ^h and A_μ^b differ at most by a pure gauge

$$A_\mu^b = A_\mu^h + \partial_\mu \phi. \quad (27)$$

Since Eq. (13) is gauge invariant when $\mathcal{M}$ is a closed manifold we conclude that

$$\mathcal{I}_{\text{CS}} = \frac{1}{8\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A_\mu^b F_{\nu\lambda}^b = \frac{1}{8\pi} \int_{\mathcal{S}^3} d^3x \epsilon^{\mu\nu\lambda} A_\mu^h F_{\nu\lambda}^h = \mathcal{H}. \quad (28)$$

Thus $\mathcal{H} = n$ implies $\mathcal{I}_{\text{CS}} = n$.

In physics one often encounters the Berry phase when a system possesses both “fast” and “slow” dynamic degrees of freedom. When the fast degrees of freedom are integrated it often produces, as part of the effective action of the slow variables, a Berry-phase term that is nonzero even when the slow variables change adiabatically with time. Such term can fundamentally alter the behavior of the slow variables.

Here we present an example where the fast degrees of freedom generate an effective action represented by the Hopf invariant of the slow variables. The model is a field theory in 2+1 space-time dimensions. It consists of two fields: (1) a fermion field $\psi_\sigma(\mathbf{r}, t)$, and (2) an unit vector field $\hat{\mathbf{n}}(\mathbf{r}, t)$. The Lagrangian density is given as

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_\psi + \mathcal{L}_n - g \hat{\mathbf{n}} \cdot \bar{\psi}_\alpha \boldsymbol{\sigma}_{\alpha\beta} \psi_\beta, \\
\mathcal{L}_\psi &= \bar{\psi}_\alpha (\partial_t - \mu) \psi_\alpha - \frac{1}{2m} \bar{\psi}_\alpha (\nabla - i \mathbf{A}_{ex})^2 \psi_\alpha, \\
\mathcal{L}_n &= i \Omega [\hat{\mathbf{n}}] + \frac{K}{2} |\nabla \hat{\mathbf{n}}|^2. \quad (29)
\end{aligned}$$

In the above m, g, c, μ are parameters of the model, $\mathbf{A}_{ex}$ is the vector potential of an external magnetic field B , i.e., $\partial_x A_y - \partial_y A_x = B$, and $\delta\Omega/\delta\hat{\mathbf{n}} = \hat{\mathbf{n}} \times \partial_\mu \hat{\mathbf{n}}$. Physically $\mathcal{L}_\psi$ describes fermions moving in an external magnetic field, and $\mathcal{L}_n$ describes the dynamics of magnetic moments in a ferromagnet. The last term in the first equation is the Zeeman coupling between the fermions and the magnetic moments. By adjusting μ we can tune the density ($\bar{\rho}$) of the fermions so that

$$\bar{\rho} = k \frac{B}{\phi_0}, \quad (30)$$

where $\phi_0 = 2\pi$ is the Dirac flux quantum and k is an integer. When Eq. (30) is satisfied, the ground state of the fermions is an “integer-quantum Hall liquid.” [9] Let us further assume that g is large so that locally the electron spins align with the direction of $\hat{n}$. Under that condition, integrating out the fer-

mion field produces a term $k/(4\pi) \int d^2x dt \epsilon^{\mu\nu\lambda} A_\mu^h \partial_\nu A_\lambda^h$, which is proportional to the Hopf invariant of the $\hat{n}(\mathbf{r}, t)$. This term has the effect of changing the spins and statistics of solitons (the skyrmions) in the $\hat{n}(\mathbf{r}, t)$ field. [7].

D.H.L. is indebted to Geoffrey Lee for helping him to visualize the dual of the Pontryagin form of the nontrivial Hopf map. We thank Qiang-Hua Wang for help in plotting the dual of the Pontryagin form. D.H.L. is supported by NSF Grant DMR 99-71503.

-
- [1] M. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
 [2] We note that in order for A_μ^b to be well defined the reference eigenvector $|\psi(\mathbf{r})\rangle$ must be differentiable with respect to $\mathbf{r}$.
 [3] P.A.M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931).
 [4] C.N. Yang, Ann. N.Y. Acad. Sci. **294**, 86 (1977).
 [5] S.S. Chern, Ann. Math. **47**, 85 (1946).

- [6] S.S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).
 [7] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
 [8] Y.-S. Wu and A. Zee, Phys. Lett. **147B**, 325 (1984).
 [9] For a review see, e.g., *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York, 1986).