Limit cycle theory of self-sustained current oscillations in sequential tunneling of superlattices

Z.Z. Sun ^a, Shi-dong Wang ^a, S.Q. Duan ^{a,b}, X.R. Wang ^a

^a*Physics Department, The Hong Kong University of Science and Technology,
Clear Water Bay, Hong Kong SAR, China*

^b*Institute of Applied Physics and Computational Mathematics, Beijing 100088,
People's Republic of China*

Abstract

A unified theory of self-sustained current oscillations (SSCOs) is presented. We establish these oscillations as the manifestations of limit cycles, around unstable steady-state solutions caused by the negative differential conductance. This theory implies that both the generation and the motion of an electric-field domain boundary are universal in the sense that they do not depend on the initial conditions. Under an extra weak ac bias with a frequency ω_{ac} , the frequency must be either ω_{ac} or an integer fractional of ω_{ac} if the tunneling current oscillates periodically in time, indicating the periodic doubling for this nonlinear dynamical system.

Key words: Superlattice; Vertical transport; Limit cycle; Nonlinear

Following the discovery of self-sustained current oscillations (SSCOs) in sequential tunneling of superlattices (SLs) under a dc bias[1], a large number of experimental and theoretical studies have focused on their origin and how these oscillations develop from steady-state solutions (SSSs). Experimentally, SSCO have been observed in both doped and undoped SLs[1]. The oscillations can be induced by continuous illumination with laser light or by a change in doping[1]. Recently, it has been shown that SSCO can also be induced by applying an external magnetic field parallel to SL layers or varying the sample temperature[2].

Our current understanding of SSCO is mainly from numerical studies[3]. Early works[3] established a correct model for SSCO. Many numerical results and some analyses were done to simulate and to reproduce many experimental results. Great progress had been made in understanding SSCO. SSCO are

understood as the motion of electric-field domains (EFDs)[4,5]. However, a simple physical picture did not appear in those early works. The understanding at the computational level is the first step, and deep insights can be only obtained when the general concepts and principles are found. A microscopic approach[6] would be accurate if all the microscopic parameters and mechanisms were known. It remains a challenge to deduce the rules of macroscopic behavior from the microscopic details. Recently, a clear route to SSCOs developed from SSSs in sequential tunneling of SLs was proposed[7]: Due to negative differential conductance (NDC)[8], an SSS is not stable. A limit cycle is generated around the unstable SSS because of the local repulsion and global attraction in the phase space. The system moves along the limit cycle, leading to an SSCO.

Unfortunately, later studies[9] show that the simple model used in Ref. [7] does not have an SSCO solution even though it gives the on-set of instability conditions of an SSS. In this paper we show that the behavior of SSCOs can be explained in terms of *limit cycle*, a fundamental concept in nonlinear science. *As manifestations of limit cycles*, the generation and motion of an EFD do not depend on the initial conditions. An EFD-boundary does not necessarily start from the emitter, and end up in the collector. Furthermore, limit cycle concept gives us a power of prediction. We predict that the generation and motion of EFD are universal, and the frequency of a periodic motion under an ac bias must be either the ac bias frequency or its integer fractional. Thus, we find that SSCOs can be understood under the general concepts of nonlinear physics.

We consider a system consisting of N quantum wells as shown schematically in Fig. 1. An external bias U is applied between the two end wells. Current flows perpendicular to the SL layers. In the sequential tunneling, charge carriers are in local equilibrium within each well, so that a chemical potential can be defined locally. The chemical potential difference between two adjacent wells is called bias V on the barrier in between. A current I_i passes through the i^{th} barrier under a given bias V_i . This current may depend on other parameters, such as doping N_D . One of the results in Ref. [7] is that an SSS must be unstable if there are two or more barriers being in the NDC regime. It is worth pointing out that, although this instability result is obtained from an analysis of a simplified sequential tunneling model, it is generally true. Without losing generality, we assume that barriers 1 and 2, separated by well 2 in Fig. 1, are under the NDC regime for an SSS. Assume the chemical potential in well 2 increases a little bit due to a fluctuation. Then bias V_1 on barrier 1 decreases while V_2 increases. Because both of the barriers are in the NDC regime, charge carriers flow more into well 2 through barrier 1 while less carriers flow out of it, leading to a further increase of the chemical potential in well 2. This drives the system away from the SSS, i.e. instability.

Fig. 1. Schematic illustration of an SL system. μ_i is the local chemical potential of the i^{th} well. μ_L and μ_R are the chemical potential of the left-hand side and right-hand side electrodes, respectively. V_i is the bias on the i th barrier, and $\mu_L - \mu_R = U$ is the external bias.

The potential barriers in SLs can be low and high, leading to so-called strongly and weakly coupled SLs. The SLs exhibiting SSCOs are normally the weakly coupled ones. For a weakly coupled SL, electrons can stay in each quantum well for a long time comparing with the tunneling time through one barrier. Different local chemical potential will be built in each well. In another word, the voltage drops occur mainly on the potential barriers. Therefore, the discretized Poisson equation shall be used to describe weakly coupled SLs. This point was also made in a recent review[10]. Following Ref. [3], the dynamics of the system is governed by the discrete Poisson equations

$$k(V_i - V_{i-1}) = n_i - N_D, \quad i = 1, 2, \dots, N \quad (1)$$

and the current continuity equations

$$J = k \frac{\partial V_i}{\partial t} + I_i, \quad i = 0, 1, 2, \dots, N \quad (2)$$

where k depends on the SL structure and its dielectric constant. In Eq. (1), N_D and n_i are electron doping and the electric charge in the i^{th} well, respectively. I_i is, in general, a function of V_i and n_i . It can be shown[9] that all SSSs are stable if I_i is a function of V_i only. On the other hand, an SSS may be unstable[3] if one chooses $I_i = n_i v(V_i)$, where v is a phenomenological drifting velocity which is, for simplicity, assumed to be a function of V_i only. The constraint equation for V_i is

$$\sum_{i=0}^N V_i = U. \quad (3)$$

Previous studies[3] proved that this model is capable of describing SSCOs. To close the equations, a proper boundary condition is needed. It is proper to assume a constant n_0 , $n_0 = \delta N_D$ with δ as a model parameter, if the carrier density in the emitter is much larger than those in wells, and its change due to a tiny tunneling current is negligible. However, it is mathematically equivalent

to other boundary conditions used in literature[3]. Our goal is to show that the limit cycle is one of the most important features in this widely studied model. We would also like to point out that instabilities can occur not only in a discrete model, but also in a continuous system. The examples include the Gunn effect for bulk semiconductors and recently demonstrated microwave induced zero-resistance state in the quantum Hall systems[11].

According to our theory, NDC is essential for SSCOs. An SSCO can only occur when there is a negative differential velocity in $v(V)$ [3]. One can assume v being a sum of a series of Lorentzian functions if NDC is due to the resonance tunneling between the discrete quasi-bounded states in wells. One may also choose a piecewise linear function in order to make an analytic investigation easy. We shall assume $v(V)$ as the sum of two Lorentzian functions, $v(V) = 0.0081/[(V/E - 1)^2 + 0.01] + 0.36/[(V/E - 2.35)^2 + 0.18]$. This v has two peaks at $V = E$ and $V = 2.35E$. A negative differential velocity exists between $V = E$ and about $V = 1.3E$. Thus, E can be used as a natural unit of bias, and $1/v(E)$ as that of the time (the lattice constant is set to be 1). For $N = 40$, $U = 43.6E$, $N_D = 0.095kE$, and $\delta = 1.001$, the set of equations has SSCO solutions. The set of equations can be solved by the Runge-Kutta method numerically. Initially, the external bias is randomly distributed. Quickly, V_i reaches a stable state *which does not depend on the initial conditions*. Fig. 2 is the projections of three bias-trajectories in $V_5 - V_{38}$ phase plane with different initial conditions. V_5 is in the low EFD while V_{38} is in the high EFD here. Clearly, one obtains a closed *isolated* curve indicating a limit cycle. The current oscillation period is the time that the system needs to move around the cycle once. It should be pointed out that, to demonstrate the existence of a limit cycle[12], *one needs to show not only the system moving on a closed curve in its phase space, but also its independence on the initial conditions*. In other words, the closed curve is isolated in the phase space. The inset is the phase diagram of SSCOs in $U - N_D$ plane with all other parameters unchanged, where N_D is in the unit of kE . The shadowed area corresponds to the SSCO regime. Of course, the diagram depends on the values of δ , N , and function of v .

Although it is known that SSCOs are accompanied by the motion of EFD boundaries, how an EFD boundary is generated and propagates inside SLs were debated[3]. According to the limit cycle picture, the charge accumulation (depletion) is responsible to the creation of an EFD boundary. Charge carriers are accumulated (depleted) in a particular well because of an imbalance of carriers flowing in and that out. This imbalance is caused by NDC as we argued early. Thus an EFD boundary can start in any well and oscillates inside an SL. Furthermore, *an EFD boundary should not depend on the initial conditions, but are completely determined by the limit cycles around unstable SSSs*.

To demonstrate the correctness of our picture, we locate numerically the

Fig. 2. Trajectories of the system in phase plane $V_5 - V_{38}$ in an SSCO regime with three different initial conditions. They reach quickly to the same isolated closed curve indicating a limit cycle in the space. The bias is in the unit of E . The inset is the phase diagram in $U-N_D$ plane with all other parameters unchanged, where N_D is in the unit of kE . The system inside the shadowed area will have an SSCO solution while it has a static current-voltage characteristic outside this area.

position of the EFD boundary in the calculation that gives Fig. 2. Fig. 3 is the evolution of the boundary. It reaches a stable oscillating state quickly. One can see that the EFD boundary oscillates between wells 26 and 37 in the SL of total 40 wells. The inset is the field distribution across the SL at points a, b, c, and d in Fig. 3. The bias V_5 (V_{38}) of the low (high) EFD moves up and down as the EFD boundary oscillates inside the SL. The stable oscillation state does not change when different initial conditions are used. In this sense both generation and motion of an EFD boundary are universal. Although early numerical calculations[3] might have already implied that an EFD boundary can start in an interior well and oscillate inside an SL, it may not be easy for theories like that of Ref. [3] to explain this universal property.

Except a few of attempts[13,14] which were mainly on the numerical aspects dealing with aperiodic time-dependent tunneling current, most theoretical studies have not considered SSCOs under the influence of an ac bias due to the lack of a clear physical picture such that one would not be able to analyze its effects. The limit cycle theory offers a way of analyzing the ac bias effect. For an SSCO in SLs, the tunneling current oscillates with an intrinsic frequency ω_0 under only a dc bias. Applying a small extra ac bias, it affects SSCOs through perturbing the system trajectories. In the case that the tunneling current can oscillate periodically with a new frequency ω , the ac bias on the system should return to its starting value after the system completes its motion on the limit cycle. It means $\omega_{ac}/\omega = n = \text{integer}$. A weak ac bias cannot greatly change the evolution trajectory of the system in the phase space. The consequences are as follows: a) Current oscillation frequency ω cannot

Fig. 3. The time evolution of an EFD boundary. The parameters are the same as those for Fig. 2. The boundary is oscillating between well 26 and well 37. The stable oscillation does not depend on the initial conditions. The inset is the electric field distribution at points a, b, c, and d in Fig. 3.

be much greater than the intrinsic frequency. Thus, at high ac bias frequency ($\omega_{ac} \gg \omega_0$), it is impossible for the bias to deform the limit cycle slightly such that the time for the system moving around the cycle once to be the same as the period of the ac bias. b) In a case that the system cannot deform itself to match ω_{ac} , a trajectory may become a closed curve after several turns in the phase space. Therefore, $\omega_0/\omega = m = \text{integer}$. Indeed, Fig. 4 is the limit cycles in phase plane V_5 - V_{38} under an extra ac bias $V_{ac} \sin(\omega_{ac}t)$ with $V_{ac} = 0.44E$ and $\omega_{ac} = 2\omega_0$ (dotted line), $3\omega_0$ (solid line) while the rest of parameters remain the same as those for Figs. 2-3. The right (left) inset is the current-time curve for $\omega_{ac} = 2\omega_0$ ($3\omega_0$) after the current oscillation becomes stable. The Fourier transformation shows the current oscillation frequency ω being ω_0 . This is the solution of $\omega_{ac}/\omega = n = \text{integer}$ and $\omega_0/\omega = m = \text{integer}$ for the smallest possible n ($=2, 3$) and m ($=1$).

For ω_{ac}/ω_0 being a non-integer rational number, ω should be different from ω_0 according to the rules of $\omega_{ac}/\omega = n = \text{integer}$ and $\omega_0/\omega = m = \text{integer}$. For example, $\omega_{ac}/\omega_0 = 1/q$ with $q = \text{integer}$, then the limit cycle can deform itself in such a way that it becomes a closed curve after q turns in the phase space, corresponding to $n = 1$ and $m = q$. In this case, the current oscillates with ω_{ac} . The solid line in Fig. 5 is the numerical results of the limit cycle in phase plane V_5 - V_{38} for $\omega_{ac} = \omega_0/3$ while the rest of parameters are kept the same as those for Figs. 2 to 4. Indeed, the limit cycle, which does not depend on the initial conditions, makes $q = 3$ turns in the phase plane as expected. The left inset is the corresponding tunneling current evolution curve. The Fourier transformation of the current evolution indeed shows $\omega = \omega_{ac}$. For $\omega_{ac}/\omega_0 = 2.5$, our rules predict the $\omega = \omega_0/2 = \omega_{ac}/5$, corresponding to $n = 5$ and $m = 2$. This result is verified by the numerical calculation as displayed

Fig. 4. The limit cycles in phase plane V_5 - V_{38} under an extra ac bias $V_{ac} \sin(\omega_{ac} t)$ with ω_{ac} being $2\omega_0$ (dotted line) and $3\omega_0$ (solid line). The inset on the right (left) is the current-time curve for $\omega_{ac} = 2\omega_0$ ($=3\omega_0$), $V_{ac} = 0.44E$, and all other parameters are the same as those for Figs. 2 and 3. The time is in the unit of $1/v(E)$. Bias is in the unit of E . The current frequencies are $\omega_{ac}/2 = \omega_0$ and $\omega_{ac}/3 = \omega_0$, respectively.

in Fig. 5 (dotted line). As before, all other parameters are kept the same as those for Figs. 2 to 4. The right inset is the corresponding tunneling current evolution curve. The limit cycle is a two-turn closed curve in phase plane V_5 - V_{38} . The Fourier transformation of the current evolution demonstrates that the current oscillates with frequency $\omega_0/2 = \omega_{ac}/5$. We would like to make following remarks. a) $\omega = \omega_{ac}$ for $\omega_{ac} = \omega_0/q$ cannot be true for all q because ω should approach ω_0 in the limit $\omega_{ac} \rightarrow 0$. b) We have considered only periodic responses of the tunneling current. *It does not rule out other more complicated behaviors.* In fact, there are reports[13–15] of quasi-periodic or chaotic current-time behaviors under a combined dc and ac biases. c) Vary external parameters, the size and shape of a limit cycle should change in general. Thus ω_0 can shift. As a consequence, $\omega_{ac}/\omega = n$ may be quite robust against ω_{ac} . When ω_{ac} is very close to ω_0 , it may be possible for the limit cycle to deform itself in such a way that ω_0 shifts to ω_{ac} . In this case, the current shall oscillate with frequency ω_{ac} . Our preliminary results indeed show so. How much ω_0 can shift depends on the magnitude of V_{ac} and ω_{ac} , as it was shown in a recent experiment[16].

We would like to emphasize that our theory is based on the limit cycle concept while other theories were based on the dynamics of electric field domain fronts. The terminology of limit cycle might be found in some publications, but it appeared only as a nice system property, not as a principle of SSCOs. Our theory is more fundamental in the sense that one can understand SSCOs in superlattices by the same principles as other nonlinear phenomena such as chemical reaction oscillations. Although numerical results are not our emphases here, we show that they can be easily understood by using this

Fig. 5. The limit cycles in phase plane $V_5 - V_{38}$ under an extra weak ac bias $V_{ac} \cos(\omega_{ac}t)$ with ω_{ac} being $\omega_0/3$ (solid line) and $2.5\omega_0$ (dotted line). The inset on the left (right) is the curve of current vs. time for $\omega_{ac} = \omega_0/3$ ($=2.5\omega_0$). All other parameters are the same as those for Fig. 4. The current frequency equals to ω_{ac} ($\omega_{ac}/5$) for $\omega_{ac} = \omega_0/3$ ($=2.5\omega_0$).

concept.

In summary, we show the existence of limit cycles, which are isolated closed curves in phase space, when SSCOs occur. SSCOs can be understood as the manifestations of limit cycles. Just like many other nonlinear dynamical systems, SSCOs are governed by the properties of the unstable SSSs. According to this theory, the generation and motion of an EFD boundary are also the properties of unstable SSSs. An EFD boundary does not necessarily need to start from the emitter. It can start from an interior well, and it then oscillates inside the SL. They are universal in the sense that they do not depend on the initial conditions. This universal property may not be so obvious in previous theories[3]. We have also investigated the effects of a small extra external ac bias on SSCOs. We find that the tunneling current will oscillate periodically when the ac bias frequency ω_{ac} is commensurate with the system intrinsic frequency ω_0 . The current frequency equals either ω_{ac} or ω_{ac}/n , where n is an integer, showing periodic doubling which is a general phenomenon in nonlinear dynamical systems.

Acknowledgement

X.R.W. would like to thank Prof. J. N. Wang and Prof. W. K. Ge for many comments. This work is supported by RGC grants, Hong Kong.

References

- [1] S.H. Kwok, T.B. Norris, L.L. Bonilla, J. Galán, J.A. Cuesta, F.C. Martínez, J.M. Molera, H.T. Grahn, K. Ploog, R. Merlin, Phys. Rev. B 51 (1995) 10171; H.T. Grahn, J. Kastrup, K. Ploog, L.L. Bonilla, J. Galán, M. Kindelan, M. Moscoso, Jpn. J. Appl. Phys. 34 (1995) 4526; J. Kastrup, R. Klann, H.T. Grahn, K. Ploog, L.L. Bonilla, J. Galán, M. Kindelan, M. Moscoso, R. Merlin, Phys. Rev. B 52 (1995) 13761.
- [2] J.N. Wang, B.Q. Sun, X.R. Wang, H.L. Wang, Solid State Commun. 112 (1999) 371; J.N. Wang, B.Q. Sun, X.R. Wang, Y.Q. Wang, W.K. Ge, H.L. Wang, Appl. Phys. Lett. 75 (1999) 2620; X.R. Wang, J.N. Wang, B.Q. Sun, D.S. Jiang, Phys. Rev. B 61 (2000) 7261.
- [3] J. Kastrup, R. Hey, K.H. Ploog, H.T. Grahn, L.L. Bonilla, M. Kindelan, M. Moscoso, A. Wacker, J. Galán, Phys. Rev. B 55 (1997) 2476; A. Wacker, M. Moscoso, M. Kindelan, L.L. Bonilla, Phys. Rev. B 55 (1997) 2466; M. Patra, G. Schwarz, and E. Schöll, Phys. Rev. B 57 (1998) 1824.
- [4] L.L. Chang, L. Esaki, R. Tsu, Appl. Phys. Lett. 24 (1974) 593; L. Esaki, L. L. Chang, Phys. Rev. Lett. 33 (1974) 495.
- [5] E. Schöll, Nonlinear spatio-temporal dynamics and chaos in semiconductors, Cambridge University Press, 2001.
- [6] A. Wacker, Phys. Rep. 357 (2002) 1.
- [7] X.R. Wang, Q. Niu, Phys. Rev. B. 59 (1999) R12755.
- [8] M. Buttiker, H. Thomas, Phys. Rev. Lett. 38 (1977) 78.
- [9] G. Xiong *et al.*, unpublished; D. Sanchez, L.L. Bonilla, G. Platero, Phys. Rev. B. 64 (2001) 115311.
- [10] L.L. Bonilla, H.T. Grahn, Rep. Prog. Phys. 68 (2005) 577.
- [11] R.G. Mani, J.H. Smet, K. von Klitzing, V. Narayanamurti, W.B. Johnson, V. Umansky, Nature 420 (2002) 646; M.A. Zudov, R.R. Du, L.N. Pfeiffer, K.W. West, Phys. Rev. Lett. 90 (2003) 046807; P.W. Anderson, W.F. Brinkman, cond-mat/0302129.
- [12] Textbooks on nonlinear physics, such as, S. H. Strogatz, Nonlinear Dynamics and Chaos: with applications to physics, biology, chemistry, and engineering, Addison-Wesley, 1994.
- [13] O.M. Bulashenko, L.L. Bonilla, Phys. Rev. B 52 (1995) 7849; O.M. Bulashenko, M.J. García, L.L. Bonilla, Phys. Rev. B 53 (1996) 10008.

- [14] A. Amann, J. Schlesner, A. Wacker, E. Schöll, Phys. Rev. B 65 (2002) 193313;
E. Schomburg, K. Hofbeck, R. Scheuerer, M. Haeussler, K.F. Renk, A.-K. Jappsen, A. Amann, A. Wacker, E. Schöll, D.G. Pavel'ev, Yu. Koschurinov, Phys. Rev. B 65 (2002) 155320.
- [15] Y.H. Zhang, R. Klann, K.H. Ploog, H.T. Grahn, Appl. Phys. Lett. 70 (1997) 2825;
K.J. Luo, S.W. Teitsworth, H. Kostial, H.T. Grahn, N. Ohtani, Appl. Phys. Lett. 74 (1999) 3845.
- [16] Z.Z. Sun, H.T. He, J.N. Wang, S.D. Wang, X.R. Wang, Phys. Rev. B 69 (2004) 045315.