Inconsistency of Bootstrap for Nonstationary, Vector Autoregressive Processes

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Abstract

Using a nonstationary, bivariate autoregressive process with iid innovations, this paper shows that the bootstrap vector autoregressive causality test is inconsistent in general in the sense that its weak limit is different from that of the original causality test.

Keywords: bootstrap, nonstationary vector autoregression, causality test

1 Introduction

Though bootstrapping has often been used for time series models, it is now well known that the bootstrap method may be inconsistent for autoregressive (AR) models with unit roots. This is documented in Basawa et al. (1991a) and Datta (1996) for the AR(1) model with a unit root and in Inoue and Kilian (2002) for the AR(p) model ($p > 1$) with a unit root.

However, consistency of the bootstrap method for nonstationary, vector autoregressive (VAR) processes has not been studied yet. In this paper, we focus on a nonstationary, bivariate AR process with iid innovations and show that the bootstrap VAR causality test is inconsistent in general in the sense that its weak limit is different from that of the original causality test. The VAR causality test, initiated by Granger (1969), has often been used in economic applications.

This paper is organized as follows. Section 2 presents the main result of this paper. Appendix contains all the proofs. All the limits are taken as $n \rightarrow \infty$.

2 Inconsistency of Bootstrap for VAR Causality Test

Suppose that the bivariate time series $\{y_t\}$ is generated by

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = A \begin{pmatrix} y_{1(t-1)} \\ y_{2(t-1)} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \quad y_{10} = y_{20} = 0, \quad (t = 1, \dots, n),$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ and $[u_{1t}, u_{2t}]' \sim iid(0, I_2)$ and $E | u_{it} |^{2+\delta} < \infty$ for $\delta > 0$ ($i = 1, 2$). In this data generation, y_{1t} and y_{2t} are independent integrated

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processes and the variance-covariance matrix of the error vector is known. The probability law for $\{(y_{1t}, y_{2t})\}$ is denoted by P . The least squares estimator of A , $\hat{A}_n$, has the limiting distribution

$$\begin{aligned} n(\hat{A}_n - I_2) &\Rightarrow \int_0^1 dW(r)W(r)' \left(\int_0^1 W(r)W(r)' dr \right)^{-1} \\ &= \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \text{ say,} \end{aligned} \quad (1)$$

where $W(r) = [W_1(r), W_2(r)]'$ is the standard vector Brownian motion of dimension two.

The null hypothesis that y_{2t} does not cause y_{1t} is

$$H_0 : a_{12} = 0, \quad (2)$$

and the Wald test for this null takes the form $\mathcal{W}_n = \hat{a}_{12n}^2 / m_{22n}$, where $\hat{a}_{12n}$ the north-west corner element of $\hat{A}_n$ and m_{22n} is the south-west corner element of $[\sum_{t=2}^n y_{t-1}y_{t-1}']^{-1}$. As reported in Toda and Phillips (1993, Theorem 2), the limiting null distribution of the Wald causality test is

$$\mathcal{W}_n \Rightarrow \left(\int_0^1 \bar{W}_2(r) dW_1(r) \right)^2 / \int_0^1 \bar{W}_2(r)^2 dr, \quad (3)$$

where $\bar{W}_2(r) = W_2(r) - \int_0^1 W_1(s)W_2(s)ds \left(\int_0^1 W_1(s)^2 ds \right)^{-1} W_1(r)$.

Let $\hat{u}_t = y_t - \hat{A}_n y_{t-1}$ where $y_t = [y_{1t}, y_{2t}]'$. The bootstrap sample $[y_{1t}^*, y_{2t}^*]'$ is obtained recursively by the relation

$$\begin{pmatrix} y_{1t}^* \\ y_{2t}^* \end{pmatrix} = \begin{bmatrix} \hat{a}_{11n} & 0 \\ \hat{a}_{21n} & \hat{a}_{22n} \end{bmatrix} \begin{pmatrix} y_{1(t-1)}^* \\ y_{2(t-1)}^* \end{pmatrix} + \begin{pmatrix} u_{1t}^* \\ u_{2t}^* \end{pmatrix}, y_{10}^* = y_{20}^* = 0, (t = 1, \dots, n),$$

where $\hat{a}_{11n}, \hat{a}_{21n}$ and $\hat{a}_{22n}$ are the elements of $\hat{A}_n$ and $[u_{1t}^*, u_{2t}^*]'$ is drawn from $\{\hat{u}_1 - \frac{1}{n} \sum_{t=1}^n \hat{u}_t, \dots, \hat{u}_n - \frac{1}{n} \sum_{t=1}^n \hat{u}_t\}$ with replacements. This data generation incorporates the null restriction (2) as recommended by Li and Maddala (1996, Subsection 2.2).

The bootstrap Wald causality test for the null hypothesis (2) is $\mathcal{W}_n^*(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n}) = \tilde{a}_{12n}^2 / m_{22n}^*$, where $\tilde{a}_{12n}$ is the least squares estimate of a_{12} using the bootstrap sample $\{[y_{1t}^*, y_{2t}^*]'\}_{t=1}^n$ and m_{22n}^* is defined in the same way as for m_{22n} replacing the original sample with the bootstrapped sample. We will show that $\mathcal{W}_n^*(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n})$'s weak limit is not that of $\mathcal{W}_n$ in (3).

Consider the triangular arrays $\{y_{1,k,n}, k \geq 1, n \geq 1\}$ and $\{y_{2,k,n}, k \geq 1, n \geq 1\}$ that are generated by

$$\begin{aligned} y_{1,k,n} &= \beta_{11n} y_{1,k-1,n} + u_{1k}^*, \quad y_{1,0,n} = 0; \\ y_{2,k,n} &= \beta_{21n} y_{1,k-1,n} + \beta_{22n} y_{2,k-1,n} + u_{2k}^*, \quad y_{2,0,n} = 0, \end{aligned} \quad (4)$$

where $\beta_{11n} = \exp(\frac{c_{11}}{n}) + o(n^{-1})$, $\beta_{21n} = \frac{c_{21}}{n} + o(n^{-1})$ and $\beta_{22n} = \exp(\frac{c_{22}}{n}) + o(n^{-1})$ with c_{11}, c_{21} and c_{22} being finite constants. The following lemma reports the weak limits of the sample moments involving $y_{1,k,n}$, $y_{2,k,n}$ and u_{1k}^* .

Lemma 1: Let $W_1(r)$ and $W_2(r)$ be independent standard Brownian motions.

- (a) $\frac{1}{n^2} \sum_{k=2}^n y_{1,k-1,n}^2 \Rightarrow^* \int_0^1 J_1(r)^2 dr$ a.s., where $J_1(r) = \int_0^r \exp[(r-s)c_{11}] dW_1(s)$;
- (b) $\frac{1}{n^2} \sum_{k=2}^n y_{2,k-1,n}^2 \Rightarrow^* \int_0^1 J_2(r)^2 dr$ a.s., where $J_2(r) = c_{21} \int_0^r \exp[(r-s)c_{22}] J_1(s) ds + \int_0^r \exp[(r-s)c_{22}] dW_2(s)$;
- (c) $\frac{1}{n^2} \sum_{k=2}^n y_{1,k-1,n} y_{2,k-1,n} \Rightarrow^* \int_0^1 J_1(r) J_2(r) dr$ a.s.;
- (d) $\frac{1}{n^2} \sum_{k=2}^n y_{1,k-1,n} u_{1k}^* \Rightarrow^* \int_0^1 J_1(r) dW_1(r)$ a.s.,

where $\Rightarrow^*$ denotes the weak convergence conditionally on $\{y_t\}_{t=1}^n$ and a.s. almost surely in P . Joint convergence of (a)-(d) also applies.

Let $\mathcal{W}_n^\dagger(\beta_{11n}, \beta_{21n}, \beta_{22n})$ denote Wald test for the null hypothesis (2) using $\{[y_{1,k,n}, y_{2,k,n}]'\}_{k=1}^n$. Lemma 1 and the continuous mapping theorem yield the weak limit of $\mathcal{W}_n^\dagger(\beta_{11n}, \beta_{21n}, \beta_{22n})$ as follows.

$$\mathcal{W}_n^\dagger(\beta_{11n}, \beta_{21n}, \beta_{22n}) \Rightarrow^* \left(\int_0^1 \bar{J}_2(r) dW_1(r) \right)^2 / \int_0^1 \bar{J}_2(r)^2 dr \text{ a.s.,}$$

where $\bar{J}_2(r) = J_2(r) - \int_0^1 J_1(s) J_2(s) ds \left(\int_0^1 J_1(s)^2 ds \right)^{-1} J_1(r)$. We denote

$$\lim_{n \rightarrow \infty} P[\mathcal{W}_n^\dagger(\beta_{11n}, \beta_{21n}, \beta_{22n}) \leq x \mid y_1, \dots, y_n] = H(c_{11}, c_{21}, c_{22}, x). \quad (5)$$

This is equivalent to the limiting distribution of $\mathcal{W}_n$ when $c_{11} = c_{21} = c_{22} = 0$.

Define the conditional probability distribution

$$H_n(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n}, x) = P[\mathcal{W}_n^*(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n}) \leq x \mid y_1, \dots, y_n]$$

and a random measure on the real line

$$\eta_n(D) = \int_D H_n(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n}, dx),$$

where D is a Borel set on the real line. Note that $\eta_n(D)$ denotes the probability of the bootstrap test $\mathcal{W}_n^*(\hat{a}_{11n}, \hat{a}_{21n}, \hat{a}_{22n})$ belonging to the Borel set D conditionally on the original data $y_1, \dots, y_n$. The latter aspect makes it a random measure. See Kallenberg (1983) for mathematical treatments on random measures.

In addition, let

$$\eta(D) = \int_D H(\gamma_{11}, \gamma_{21}, \gamma_{22}, dx),$$

where γ 's are defined in relation (1) and $H(\cdot, \cdot, \cdot, \cdot)$ in (5). $\eta(D)$ denotes the limiting probability of $\mathcal{W}_n^\dagger(\gamma_{11}, \gamma_{21}, \gamma_{22})$ belonging to the Borel set D . Since γ 's are random variables, $\eta(D)$ is a random measure.

The main result of this paper is stated in the following theorem.

Theorem 1: As $n \rightarrow \infty$,

$$\eta_n \Rightarrow \eta$$

in the space of probability measures on the real line topologized by weak convergence.

Note that $\eta(D) = \lim_{n \rightarrow \infty} P\{\mathcal{W}_n \in D\}$ if and only if $\gamma_{11} = \gamma_{21} = \gamma_{22} = 0$ *a.s.* But the probability of the latter event is zero since γ 's have continuous distributions. Thus, the above theorem shows that the bootstrap Wald causality test is inconsistent in the sense that its weak limit is different from that of the original Wald test almost surely. The subsampling method of Politis, Romano and Wolf (1999) can correct the situation as studied in Choi (2005).

Appendix: Proofs

PROOF OF LEMMA 1: Consider the triangular arrays $\{x_{1,k,n}, k \geq 1, n \geq 1\}$ and $\{x_{2,k,n}, k \geq 1, n \geq 1\}$ that are generated by

$$\begin{aligned} x_{1,k,n} &= \exp\left(\frac{c_{11}}{n}\right)x_{1,k-1,n} + u_{1k}^*, \quad x_{1,0,n} = 0; \\ x_{2,k,n} &= \left(\frac{c_{21}}{n}\right)x_{1,k-1,n} + \exp\left(\frac{c_{22}}{n}\right)x_{2,k-1,n} + u_{2k}^*, \quad x_{2,0,n} = 0 \end{aligned}$$

with u_{1k}^* and u_{2k}^* being the same as in (4). Let $z_{1,k,n} = y_{1,k,n} - x_{1,k,n}$. If

$$\sup_{0 \leq r \leq 1} |n^{-1/2} z_{1,[nr],n}| \xrightarrow{p^*} 0 \text{ a.s.}, \quad (6)$$

where $[nr]$ is the integer part of nr and $\xrightarrow{p^*}$ denotes convergence in probability given $\{y_t\}_{t=1}^n$, we have

$$n^{-1/2} y_{1,[nr],n} \Rightarrow^* J_1(r) \text{ a.s.} \quad (7)$$

because

$$n^{-1/2} x_{1,[nr],n} \Rightarrow^* J_1(r) \text{ a.s.} \quad (8)$$

which is deduced from the bootstrap invariance principle (cf. Basawa, Mallik, McCormick, Reeves and Taylor, 1991b and Feretti and Romo, 1996) and Lemma 1 of Phillips (1987). Let $z_{2,k,n} = y_{2,k,n} - x_{2,k,n}$. If

$$\sup_{0 \leq r \leq 1} |n^{-1/2} z_{2,[nr],n}| \xrightarrow{p^*} 0 \text{ a.s.} \quad (9)$$

and

$$n^{-1/2} x_{2,[nr],n} \Rightarrow^* J_2(r) \text{ a.s.}, \quad (10)$$

we have

$$n^{-1/2} y_{2,[nr],n} \Rightarrow^* J_2(r) \text{ a.s.} \quad (11)$$

If (7) and (11) hold, the stated results can be proven straightforwardly in the same manner as in Phillips (1987). Thus, the proof will be completed once we prove relations (6), (9) and (10). Lemma A.3 at the end of this appendix proves these, so the stated results follow. ■

PROOF OF THEOREM 1: By the Skorohod representation theorem (see, e.g., Davidson, 1994, Section 26.6), we may define $\tilde{a}_{ijn}$ and $\tilde{\gamma}_{ij}$ such that $\hat{a}_{ijn} \stackrel{d}{=} \tilde{a}_{ijn}$, $\gamma_{ij} \stackrel{d}{=} \tilde{\gamma}_{ij}$, and

$$n(\tilde{a}_{11n} - 1) \rightarrow \tilde{\gamma}_{11} \text{ a.s.}; n\tilde{a}_{21n} \rightarrow \tilde{\gamma}_{21} \text{ a.s.}; n(\tilde{a}_{22n} - 1) \rightarrow \tilde{\gamma}_{22} \text{ a.s.},$$

which can be rewritten as

$$\tilde{a}_{11n} = \exp(\tilde{\gamma}_{11}/n) + o_{a.s.}(n^{-1}); \tilde{a}_{21n} = \tilde{\gamma}_{21}/n + o_{a.s.}(n^{-1}); \tilde{a}_{22n} = \exp(\tilde{\gamma}_{22}/n) + o_{a.s.}(n^{-1}). \quad (12)$$

Relations (5) and (12) imply

$$H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, x) \rightarrow H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, x) \text{ a.s.} \quad (13)$$

Let I_i 's be disjoint intervals on the real line. Then, (13) implies

$$\begin{aligned} & (H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, I_1), \dots, H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, I_k)) \\ & \rightarrow (H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, I_1), \dots, H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, I_k)) \text{ a.s.} \end{aligned}$$

Here $H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, I_j) = P[\mathcal{W}_n^*(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}) \in I_j \mid y_1, \dots, y_n]$ and $H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, I_1) = \lim_{n \rightarrow \infty} P[\mathcal{W}_n^\dagger(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}) \in I_j \mid (j = 1, \dots, k)]$. Using the relations

$$\begin{aligned} & (\eta_n(I_1), \dots, \eta_n(I_k)) \stackrel{d}{=} (H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, I_1), \dots, H_n(\tilde{a}_{11n}, \tilde{a}_{21n}, \tilde{a}_{22n}, I_k)), \\ & (\eta(I_1), \dots, \eta(I_k)) \stackrel{d}{=} (H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, I_1), \dots, H(\tilde{\gamma}_{11}, \tilde{\gamma}_{21}, \tilde{\gamma}_{22}, I_k)) \end{aligned}$$

and applying Theorem 4.2 of Kallenberg (1983) yield the required result. ■

The proof of Lemma A.3 uses the following two lemmas.

Lemma A1: For $x, y, m \geq 0$,

$$[\exp(x) + y]^{2m} \leq \exp[2m(x + y)].$$

PROOF: Omitted. ■

Lemma A2: For a $k \times k$ matrix $W = (w_{ij})$, let $\|W\| = \max_{1 \leq i \leq k} \sum_{j=1}^k |w_{ij}|$.

$$(a) \frac{1}{n} \sum_{t=1}^n \hat{u}_t \rightarrow 0 \text{ a.s. where } \hat{u}_t = y_t - \hat{A}_n y_{t-1};$$

$$(b) \frac{1}{n} \sum_{t=1}^n \hat{u}_t \hat{u}_t' \rightarrow I_2 \text{ a.s.}$$

PROOF: Since $y_{lt} = \sum_{i=1}^t u_{li}$ ($l = 1, 2$), due to Strassen's (1964) law of the iterated logarithm,

$$\sum_{t=2}^n y_{l(t-1)} = O_{a.s.}(n^{3/2} \sqrt{\ln \ln(n)}); \quad (14)$$

$$\sum_{t=2}^n y_{l(t-1)}^2 = O_{a.s.}(n^2 \ln \ln(n)); \quad (15)$$

$$\sum_{t=2}^n y_{1(t-1)} y_{2(t-1)} = O_{a.s.}(n^2 \ln \ln(n)). \quad (16)$$

Moreover,

$$\sum_{t=2}^n y_{l(t-1)} u_{mt} = o_{a.s.} \left(\left(\sum_{t=2}^n y_{l(t-1)}^2 \right)^{1/2} \ln \left(\sum_{t=2}^n y_{l(t-1)}^2 \right) \right) \quad (m = 1, 2) \quad (17)$$

as shown in Lai and Wei (1982, Lemma 2 (iii)). Using the relation

$$\begin{aligned} \|\hat{A}_n - I_2\| &\leq \left\| \sum_{t=2}^n u_t y'_{t-1} \right\| \left\| \left(\sum_{t=2}^n y_{t-1} y'_{t-1} \right)^{-1} \right\| \\ &= \max \left(\left| \sum_{t=2}^n u_{1t} y_{1(t-1)} \right| + \left| \sum_{t=2}^n u_{1t} y_{2(t-1)} \right|, \left| \sum_{t=2}^n u_{2t} y_{1(t-1)} \right| + \left| \sum_{t=2}^n u_{2t} y_{2(t-1)} \right| \right) \\ &\quad \times \max \left(\left| \sum_{t=2}^n y_{2(t-1)}^2 \right| + \left| \sum_{t=2}^n y_{1t} y_{2(t-1)} \right|, \left| \sum_{t=2}^n y_{1(t-1)}^2 \right| + \left| \sum_{t=2}^n y_{1t} y_{2(t-1)} \right| \right) \\ &\quad \div \left(\sum_{t=2}^n y_{2(t-1)}^2 \sum_{t=2}^n y_{1(t-1)}^2 - \left(\sum_{t=2}^n y_{1t} y_{2(t-1)} \right)^2 \right) \end{aligned}$$

along with (15), (16) and (17), we obtain

$$\|\hat{A}_n - I_2\| = o_{a.s.} \left(\frac{\ln(n)}{n \sqrt{\ln \ln(n)}} \right). \quad (18)$$

Write

$$\frac{1}{n} \sum_{t=1}^n \hat{u}_t = \frac{1}{n} \sum_{t=1}^n u_t - \frac{1}{n} (\hat{A}_n - I_2) \sum_{t=2}^n y_{t-1}.$$

But

$$\begin{aligned} \frac{1}{n} \left\| (\hat{A}_n - I_2) \sum_{t=2}^n y_{t-1} \right\| &\leq \frac{1}{n} \|\hat{A}_n - I_2\| \left\| \sum_{t=2}^n y_{t-1} \right\| \\ &= o_{a.s.} \left(\frac{\ln(n)}{\sqrt{n}} \right) \end{aligned} \quad (19)$$

due to (18) and (14). Since $\frac{1}{n} \sum_{t=1}^n u_t \rightarrow 0$ *a.s.* by the strong law of large numbers, the stated result follows from (19). ■

(b) We have

$$\frac{1}{n} \sum_{t=1}^n \hat{u}_t \hat{u}'_t = \frac{1}{n} \sum_{t=1}^n u_t u'_t - \frac{1}{n} \sum_{t=2}^n u_t y'_{t-1} \left(\sum_{t=2}^n y_{t-1} y'_{t-1} \right)^{-1} \sum_{t=2}^n y_{t-1} u'_t.$$

(18) and (17) yield

$$\begin{aligned}
& \frac{1}{n} \left\| \sum_{t=2}^n u_t y'_{t-1} \left(\sum_{t=2}^n y_{t-1} y'_{t-1} \right)^{-1} \sum_{t=2}^n y_{t-1} u'_t \right\| \\
&= \frac{1}{n} \left\| (\hat{A}_n - I_2) \sum_{t=2}^n y_{t-1} u'_t \right\| \\
&\leq \frac{1}{n} \left\| (\hat{A}_n - I_2) \right\| \left\| \sum_{t=2}^n y_{t-1} u'_t \right\| \\
&= o_{a.s.} \left(\frac{(\ln(n))^{3/2}}{n} \right),
\end{aligned}$$

and $\frac{1}{n} \sum_{t=1}^n u_t u'_t \rightarrow I_2$ a.s. by the strong law of large numbers. Thus, the stated result follows. ■

Lemma A3: (a) $\sup_{0 \leq r \leq 1} |n^{-1/2} z_{1,[nr],n}| \xrightarrow{p^*} 0$ a.s.;

(b) $n^{-1/2} x_{2,[nr],n} \Rightarrow^* J_2(r)$ a.s.;

(c) $\sup_{0 \leq r \leq 1} |n^{-1/2} z_{2,[nr],n}| \xrightarrow{p^*} 0$ a.s.

PROOF: (a) Let $\beta_{11n} = \exp(\frac{c_{11}}{n}) + \frac{d_{11n}}{n}$ with $d_{11n} = o(1)$. Then,

$$\begin{aligned}
z_{1,k,n} &= \exp(\frac{c_{11}}{n}) z_{1,k-1,n} + \frac{d_{11n}}{n} y_{1,k-1,n} \\
&= d_{11n} \sum_{j=1}^k n^{-1} \exp \left[(k-j) \frac{c_{11}}{n} \right] y_{1,j-1,n},
\end{aligned}$$

giving

$$\begin{aligned}
& \sup_{0 \leq r \leq 1} |n^{-1/2} z_{1,[nr],n}| \\
&= \sup_{0 \leq r \leq 1} |d_{11n} \sum_{j=1}^{[nr]} n^{-1} \exp \left[([nr] - j) \frac{c_{11}}{n} \right] n^{-1/2} y_{1,j-1,n}| \\
&\leq |d_{11n}| \sup_{0 \leq r \leq 1} \left(\max_{1 \leq j \leq [nr]} |n^{-1/2} y_{1,j,n}| \right) \sum_{j=1}^{[nr]} n^{-1} \exp \left[([nr] - j) \frac{c_{11}}{n} \right] \\
&= |d_{11n}| \left(\max_{1 \leq j \leq n} |n^{-1/2} y_{1,j,n}| \right) \sup_{0 \leq r \leq 1} \sum_{j=1}^{[nr]} n^{-1} \exp \left[([nr] - j) \frac{c_{11}}{n} \right].
\end{aligned}$$

Since $d_{11n} = o(1)$ and

$$\sup_{0 \leq r \leq 1} \sum_{j=1}^{[nr]} n^{-1} \exp \left[([nr] - j) \frac{c_{11}}{n} \right] \rightarrow \sup_{0 \leq r \leq 1} \int_0^r \exp[(r-s)c_{11}] ds < \infty,$$

the result follows if

$$\max_{1 \leq j \leq n} |n^{-1/2} y_{1,j,n}| = O_{P^*}(1) \text{ a.s.}, \quad (20)$$

where P^* denotes the probability measure given $\{y_t\}_{t=1}^n$. Write

$$y_{1,j,n} = \sum_{i=1}^j \left[\exp\left(\frac{c_{11}}{n}\right) + \frac{d_{11n}}{n} \right]^{j-i} u_{1,i}^*.$$

By the Kolmogorov maximal inequality, for $\Xi > 0$,

$$P^*\left\{\max_{1 \leq j \leq n} |n^{-1/2} y_{1,j,n}| \geq \Xi\right\} \leq \frac{1}{\Xi^2} n^{-1} \text{Var}^*(y_{1,n,n}). \quad (21)$$

But

$$\begin{aligned} n^{-1} \text{Var}^*(y_{1,n,n}) &= n^{-1} \sum_{i=1}^n \left[\exp\left(\frac{c_{11}}{n}\right) + \frac{d_{11n}}{n} \right]^{2(n-i)} E^*(u_{11}^{*2}) \\ &\leq n^{-1} \sum_{i=1}^n \left[\exp\left(\frac{|c_{11}|}{n}\right) + \frac{|d_{11n}|}{n} \right]^{2(n-i)} E^*(u_{11}^{*2}) \\ &\leq n^{-1} \sum_{i=1}^n \exp\left[2(|d_{11n}| + |c_{11}|) \frac{n-i}{n}\right] E^*(u_{11}^{*2}) \\ &= \left(\sum_{i=1}^n \int_{(i-1)/n}^{i/n} \exp[2(|d_{11n}| + |c_{11}|)(1-s)] ds + o(1) \right) E^*(u_{11}^{*2}) \\ &= \left(\int_0^1 \exp[2(|d_{11n}| + |c_{11}|)(1-s)] ds + o(1) \right) E^*(u_{11}^{*2}). \quad (22) \end{aligned}$$

Note that the second inequality utilizes Lemma A.1. The first term in the last part of relation (22) converges to $\int_0^1 \exp[2|c_{11}|(1-s)] ds < \infty$ due to the dominated convergence theorem. Lemma A.2 gives

$$E^*(u_{11}^{*2}) = \frac{1}{n} \sum_{t=1}^n \hat{u}_{1t}^2 - \left(\frac{1}{n} \sum_{t=1}^n \hat{u}_{1t}\right)^2 \rightarrow 1 \text{ a.s.}$$

Thus,

$$n^{-1} \text{Var}^*(y_{1,n,n}) \rightarrow \int_0^1 \exp[2|c_{11}|(1-s)] ds \text{ a.s.}$$

This and (21) allow us to choose Ξ for every $\varepsilon > 0$ such that $P^*\{\max_{1 \leq j \leq n} |n^{-1/2} y_{1,j,n}| \geq \Xi\} \leq \varepsilon$ for all $n \geq N_\varepsilon$ with N_ε being an integer that depends on ε , which implies (20). This completes the proof. ■

(b) Write

$$x_{2,k,n} = \sum_{j=1}^k \exp\left[(k-j) \frac{c_{22}}{n}\right] \left[\frac{c_{21}}{n} x_{1,j-1,n} + u_{2j}^*\right]$$

and thus

$$\begin{aligned} x_{2,[nr],n} &= \sum_{j=1}^{[nr]} \exp \left[([nr] - j) \frac{c_{22}}{n} \right] \frac{c_{21}}{n} x_{1,j-1,n} + \sum_{j=1}^{[nr]} \exp \left[([nr] - j) \frac{c_{22}}{n} \right] u_{2j}^* \\ &= F_{1n} + F_{2n}, \text{ say.} \end{aligned}$$

Let

$$R_n(r) = n^{-1/2} x_{1,[nr],n} = n^{-1/2} x_{1,j-1,n}, \quad \frac{(j-1)}{n} \leq r < \frac{j}{n}, \quad (j = 1, \dots, n).$$

Relation (8) and the continuous mapping theorem give

$$\begin{aligned} n^{-1/2} F_{1n} &= c_{21} \sum_{j=1}^{[nr]} \exp \left[([nr] - j) \frac{c_{22}}{n} \right] \int_{(j-1)/n}^{j/n} R_n(s) ds \\ &= c_{21} \sum_{j=1}^{[nr]} \int_{(j-1)/n}^{j/n} \exp [(r-s)c_{22}] R_n(s) ds + o_p(1) \\ &= c_{21} \int_0^r \exp [(r-s)c_{22}] R_n(s) ds + o_p(1) \\ &\Rightarrow * c_{21} \int_0^r \exp [(r-s)c_{22}] J_1(s) ds \text{ a.s.} \end{aligned}$$

In addition, using the bootstrap invariance principle and the same method as in Lemma 1 of Phillips (1987), we obtain $n^{-1/2} F_{2n} \Rightarrow^* \int_0^r \exp [(r-s)c_{22}] dW_2(s)$ a.s. Thus, the stated result follows. ■

(c) This can be proved using the same methods as for part (a). Thus, details are omitted. ■

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