A DETERMINANT CONDITION FROM COMPLEX OSCILLATION THEORY

Kin Y. Li

Department of Mathematics
Hong Kong University of Science and Technology, Hong Kong

Abstract. We give an explicit description of the solutions to the determinant condition on a third order complex linear differential equation studied by Chiang, Laine and Wang.

1. INTRODUCTION Content-Length: 12564

For complex differential equations, there is an interesting line of investigations on the distribution of the roots of the solutions. If a complex-valued function \( f(z) \) has roots \( z_1, z_2, \ldots \) (ordered by nondecreasing moduli), the exponent of convergence \( \lambda(f) \) of \( f \) is defined as \( \inf \{ \lambda : \sum 1/|z_j|^{\lambda} < \infty \} \). The oscillation theory of complex differential equations (see [1] and [4]) investigates the exponents of convergence of the solutions, which yield informations on the distributions of the roots.

In [2], Bank and Laine studied the oscillation theory of certain second order differential equations with entire periodic coefficients and found explicit representations for the solutions with finite exponents of convergence. Recently, in [3], Chiang, Laine and Wang established the following result concerning the oscillation theory of a third order linear differential equation with entire periodic coefficients using Nevanlinna value distribution theory.

Theorem. Let \( K \in \mathbb{C} \) and suppose that

\[
f'''' - K f' + e^z f = 0
\]

admits a non-trivial solution \( f \) such that

\[
\log^+ N(r, 1/f) = o(r) \quad \text{as } r \to \infty.
\]

Then there exist integers \( r \) and \( s \) such that \( r + s \geq 0 \) and

\[
K = \frac{(r + s + 1)^2}{9}.
\]
Moreover, if \( n = r + s > 0 \), then \( n \) satisfies the following tridiagonal \((n+1) \times (n+1)\)-determinant condition:

\[
\det A = 0,
\]

where the non-zero diagonals of \( A \) are determined by

\[
\begin{align*}
a_{j,j-1} &:= (j-1)j(j+1) - 2jn - j^2, & j = 1, \ldots, n, \\
a_{j,j} &:= -3j(j+1) + 2n + n^2, & j = 0, \ldots, n, \\
a_{j,j+1} &:= 3(j+1), & j = 0, \ldots, n - 1.
\end{align*}
\]

Furthermore, \( f \) admits one of the following representations:

\[
f_i(z) = e^{-(s+1)z/3} \psi(e^{z/3}) \exp(c_i e^{z/3}),
\]

where \( c_i^3 + 27 = 0, i = 1, 2, 3 \), and

\[
\psi(\zeta) = \sum_{j=-r}^{s} d_j \zeta^j, \quad d_{-r}d_s \neq 0.
\]

Conversely, suppose \( K \) takes the form (1.3) and, if \( n = r + s > 0 \) and \( n \) satisfies (1.4) and (1.5), then there exist a rational function of the form (1.7) such that the three functions defined by (1.6) are linearly independent solutions of (1.1) each with exponent of convergence \( \lambda(f_i) \leq 1 \) for \( i = 1, 2, 3 \).

As observed in the concluding remarks of [3], the matrix \( A \) can be expressed as the product of three matrices \((\beta_{ij}), (\alpha_{ij})\) and \((\gamma_{ij})\), where \((\alpha_{ij})\) is a diagonal matrix such that

\[
\alpha_{0,0} = a_{0,0} = n(n+2)
\]

and

\[
\alpha_{j+1,j+1} = a_{j+1,j+1} - \frac{a_{j+1,j}a_{j,j+1}}{a_{j,j}} = A_j + \frac{B_j}{\alpha_{j,j}}, \quad j = 0, \ldots, n - 1,
\]

with

\[
A_j = a_{j+1,j+1} = n(n+2) - 3(j+1)(j+2)
\]

and

\[
B_j = -a_{j+1,j}a_{j,j+1} = 3(j+1)^2[n(n+2) - j(j+2)],
\]

while \((\beta_{ij})\) is a lower triangular matrix such that \( \beta_{jj} = 1 \) for all \( j \), and \((\gamma_{ij})\) is an upper triangular matrix such that \( \gamma_{jj} = 1 \) for all \( j \). So the determinant condition (1.4) is equivalent to the product of \( \alpha_{j,j}, j = 0, \ldots, n \) being zero.

In [3], Chiang, Laine and Wang expressed that the determinant condition (1.4) in the theorem seemed to be equivalent to \( n \not\equiv 2 \pmod{3} \), which also meant \( K \) would be a ninth of a perfect square, but not an integer. Here, we will prove that this is indeed the case. In the next section, we will first give the proof of the easier half, namely the determinant condition implies \( n \not\equiv 2 \pmod{3} \). Then the remaining half will be established by analyzing the entries \( \alpha_{j,j} \) using three recurrence relations.
2. PROOF OF EQUIVALENCE

First we present a proof of the easier half.

Proposition 2.1. If \( \det A = \prod_{j=0}^{n} \alpha_{j,j} = 0 \), then \( n \not\equiv 2 \pmod{3} \).

Proof. Suppose \( n \equiv 2 \pmod{3} \), then \( n(n + 2) \equiv -1 \pmod{3} \). We will show by induction that the entries \( \alpha_{j,j} \) is either of the form \((3p_j + 1)/(3q_j - 1)\) or of the form \((3p_j - 1)/(3q_j + 1)\), where \( p_j \) and \( q_j \) are integers. Then we will get the contradiction

\[
\det A = \prod_{j=0}^{n} \alpha_{j,j} = \prod_{j=0}^{n} \frac{3p_j \pm 1}{3q_j \mp 1} \neq 0.
\]

Now since \( \alpha_{0,0} = n(n + 2) \equiv -1 \pmod{3} \), \( \alpha_{0,0} = (3p_0 - 1)/1 \). For the inductive step, suppose \( \alpha_{j,j} = (3p_j + 1)/(3q_j - 1) \). Since \( A_j \equiv -1 \pmod{3} \) and \( B_j \equiv 0 \pmod{3} \), we have \( A_j = 3r_j + 1 \) and \( B_j = 3s_j \) for some integers \( r_j, s_j \) and

\[
\alpha_{j+1,j+1} = A_j + \frac{B_j}{\alpha_{j,j}} = \frac{(3r_j - 1) + \frac{3s_j}{(3p_j + 1)/(3q_j - 1)}}{3p_j + 1} = \frac{3(r_j p_j - p_j + r_j - 3s_j q_j - s_j) - 1}{3p_j + 1}.
\]

Similarly, suppose \( \alpha_{j,j} = (3p_j - 1)/(3q_j + 1) \), then

\[
\alpha_{j+1,j+1} = A_j + \frac{B_j}{\alpha_{j,j}} = \frac{3(r_j p_j - p_j - r_j + 3s_j q_j + s_j) + 1}{3p_j - 1}.
\]

This completes the induction. \( \square \)

Next, we will examine the converse.

Proposition 2.2. The sequences \( \{f_k\}, \{g_k\}, \{h_k\} \) defined by

\[
(2.1) \quad f_0(x) = g_0(x) = h_0(x) = 1
\]

and for \( k = 1, 2, 3, \ldots \),

\[
(2.2) \quad f_k(x) = [x - 3(3k - 1)3k]h_{k-1}(x) + 3(3k - 1)^2g_{k-1}(x),
\]

\[
(2.3) \quad [x - 3k(3k + 2)]g_k(x) = [x - 3(3k)(3k + 1)]f_k(x)
+ 27k^2[x - (3k - 1)(3k + 1)]h_{k-1}(x),
\]

\[
(2.4) \quad [x - (3k + 1)(3k + 2)]h_k(x) = [x - 3(3k + 1)(3k + 3)]g_k(x) + 3(3k + 1)^2f_k(x)
\]

are all polynomials.

Assuming the truth of Proposition 2.2 for the moment, the converse of Proposition 2.1 follows easily.
Theorem 2.3. Let \( m = n(n + 2) \), then

\[
\alpha_{j,j} = \begin{cases} 
\frac{[m - 3k(3k + 2)]g_k(m)}{f_k(m)} & \text{if } j = 3k, \\
\frac{[m - (3k + 1)(3k + 3)]h_k(m)}{g_k(m)} & \text{if } j = 3k + 1, \\
\frac{f_{k+1}(m)}{h_k(m)} & \text{if } j = 3k + 2.
\end{cases}
\]

Consequently,

\[
\det A = \prod_{j=0}^{n} \alpha_{j,j} = \begin{cases} 
g_l(m) \times \prod_{j=0}^{n} [m - j(j + 2)] = 0 & \text{if } n = 3l, \\
h_l(m) \times \prod_{j=0}^{n} [m - j(j + 2)] = 0 & \text{if } n = 3l + 1, \\
f_{l+1}(m) \times \prod_{j=0}^{n} [m - j(j + 2)] \neq 0 & \text{if } n = 3l + 2.
\end{cases}
\]

Proof of Theorem 2.3. The three initial cases \( \alpha_{0,0} = m, \alpha_{1,1} = m - 3, \alpha_{2,2} = m - 6 \) follow easily from (2.1) and (2.2). Next, suppose the case \( j = 3k - 1 \) is true, then using (2.3), we have

\[
\alpha_{3k,3k} = A_{3k-1} + \frac{B_{3k-1}}{\alpha_{3k-1,3k-1}} = m - 3(3k)(3k + 1) + 3(3k)^2[m - (3k - 1)(3k + 1)] \frac{h_{k-1}(m)}{f_k(m)} = [m - 3k(3k + 2)] \frac{g_k(m)}{f_k(m)}.
\]

Suppose the case \( j = 3k \) is true, then using (2.4), we have

\[
\alpha_{3k+1,3k+1} = A_{3k} + \frac{B_{3k}}{\alpha_{3k,3k}} = m - 3(3k + 1)(3k + 2) + \frac{3(3k + 1)^2f_k(m)}{g_k(m)} = [m - (3k + 1)(3k + 3)] \frac{h_k(m)}{g_k(m)}.
\]

Suppose the case \( j = 3k + 1 \) is true, then using (2.2), we have

\[
\alpha_{3k+2,3k+2} = A_{3k+1} + \frac{B_{3k+1}}{\alpha_{3k+1,3k+1}} = m - 3(3k + 2)(3k + 3) + \frac{3(3k + 2)^2g_k(m)}{h_k(m)} = \frac{f_{k+1}(m)}{h_k(m)}.
\]
Therefore, the first statement follows by induction. As for the last statement, it follows from telescoping products, the factor \( j = n \) and Proposition 2.1.

To complete the overall argument, we now give a proof of Proposition 2.2.

**Proof of Proposition 2.2.** Suppose \( f_j, g_j, h_j \) are polynomials for \( j = 0, 1, \ldots, k - 1 \). Then (2.2) implies \( f_k \) is a polynomial. Now to show \( g_k \) is a polynomial, it is equivalent to showing the right side of (2.3) vanishes at \( x = 3k(3k + 2) \). Substituting \( x_0 = 3k(3k + 2) \) for \( x \) into the right side of (2.3) and simplifying, we see that we have to show

\[
(2.5) \quad f_k(x_0) = 9kh_{k-1}(x_0).
\]

Now by (2.2), we get

\[
(2.6) \quad f_k(x_0) = -3k(6k - 5)h_{k-1}(x_0) + 3(3k - 1)^2g_{k-1}(x_0).
\]

By (2.4) with \( k \) replaced by \( k - 1 \), we get (after cancelling a factor of 3)

\[
(2.7) \quad 4k h_{k-1}(x_0) = (3k - 2)^2f_{k-1}(x_0) - (6k^2 - 11k + 2)g_{k-1}(x_0).
\]

Substituting (2.7) into (2.6), we get

\[
(2.8) \quad 4f_k(x_0) = 3(36k^3 - 60k^2 + 43k - 6)g_{k-1}(x_0) - 3(6k - 5)(3k - 2)^2f_{k-1}(x_0).
\]

Next if in (2.2) we replace \( k \) by \( k + 1 \), multiply both sides by \( x - (3k + 1)(3k + 2) \), and apply (2.4) to the right side, we will get (after grouping terms)

\[
(2.9) \quad [x - (3k + 1)(3k + 3)]f_{k+1}(x) = [x^2 - 3(3k + 2)^2x + 6(3k + 1)(3k + 2)^2(3k + 3)]g_k(x) + 3(3k + 1)^2[x - 3(3k + 2)(3k + 3)]f_k(x).
\]

If we replace \( k \) by \( k - 1 \) in (2.4) and apply it to the right side of (2.3), we get

\[
(2.10) \quad [x - 3k(3k + 2)]g_k(x) = [x - 9k(3k + 1)]f_k(x) + \frac{27k^2[x - (3k - 1)(3k + 1)]}{x - (3k - 2)(3k)} \times \left\{ [x - 3(3k - 2)(3k - 1)]g_{k-1}(x) + 3(3k - 2)^2f_{k-1}(x) \right\}.
\]

Observe that (2.9) allows us to express \( g_k \) in terms of \( f_k \) and \( f_{k+1} \). Replacing \( k \) by \( k - 1 \), we can express \( g_{k-1} \) in terms of \( f_{k-1} \) and \( f_k \). In (2.10), if we substitute \( g_k \) in terms of \( f_k \) and \( f_{k+1} \) and \( g_{k-1} \) in terms of \( f_{k-1} \) and \( f_k \), then we will get a recurrence relation involving \( f_{k-1}, f_k \) and \( f_{k+1} \). Setting \( x = x_0 = 3k(3k + 2) \) in this recurrence relation and simplifying, we will get

\[
(2.11) \quad 2k(6k - 5)f_k(x_0) = 9(3k - 1)(3k - 2)^2f_{k-1}(x_0).
\]

If we solve for \( f_{k-1}(x_0) \) in (2.11) and substitute that expression into the right side of (2.8), then after simplification, we will get

\[
(2.12) \quad 9(3k - 1)g_{k-1}(x_0) = 2f_k(x_0).
\]

If we solve for \( g_{k-1}(x_0) \) in (2.12) and substitute that expression into the right side of (2.6), then after simplification, we will get (2.5). Therefore, \( g_k \) is a polynomial.
Now to show \( h_k \) is a polynomial, it is equivalent to showing the right side of (2.4) vanishes at \( x = (3k + 1)(3k + 3) \). Substituting \( x_1 = (3k + 1)(3k + 3) \) for \( x \) into the right side of (2.4), we see that we have to show

\[(2.13) \quad (2k + 1)g_k(x_1) = (3k + 1)f_k(x_1).\]

Now by (2.2), we get

\[(2.14) \quad f_k(x_1) = -3(6k^2 - 7k - 1)h_{k-1}(x_1) + 3(3k - 1)^2g_{k-1}(x_1).\]

By (2.3), we get (after cancelling a factor of 3)

\[(2.15) \quad (2k + 1)g_k(x_1) = -(2k - 1)(3k + 1)f_k(x_1) + 36k^2(3k + 1)h_{k-1}(x_1).\]

Next if we use (2.2) to substitute \( f_k(x) \) in (2.4), we will get

\[(2.16) \quad [x - (3k + 1)(3k + 3)]h_k(x) = [x - 3(3k + 1)(3k + 2)]g_k(x) + [x - 3(3k - 1)(3k)]h_{k-1}(x) + 3(3k - 1)^2g_{k-1}(x).\]

If we use (2.2) to substitute away \( f_k(x) \) in (2.3), then after simplification, we will get

\[(2.17) \quad [x^2 - 27k^2x + 54k^2(9k^2 - 1)]h_{k-1}(x) = [x - 3k(3k + 2)]g_k(x) - 3(3k - 1)^2[x - 9k(3k + 1)]g_{k-1}(x).\]

Replacing \( k \) by \( k + 1 \) in (2.17), we can express \( h_k \) in terms of \( g_k \) and \( g_{k+1} \). In (2.16), if we substitute \( h_{k-1} \) using (2.17) and \( h_k \) using the “\( k \) replaced by \( k + 1 \)” version of (2.17), we will get a recurrence relation involving \( g_{k-1}, g_k \) and \( g_{k+1} \). Setting \( x = x_1 = (3k + 1)(3k + 3) \) in this recurrence relation and simplifying, we will get

\[(2.18) \quad (4k^2 - 1)g_k(x_1) = 18k(3k - 1)^2g_{k-1}(x_1).\]

If we solve for \( g_{k-1}(x_1) \) in (2.8) and substitute that expression into (2.14), we will get

\[(2.19) \quad f_k(x_1) = -3(6k^2 - 7k - 1)h_{k-1}(x_1) + \frac{4k^2 - 1}{6k}g_k(x).\]

Multiplying both sides of (2.15) by \( 2k - 1 \) and making substitution to the right side of (2.19), we will get (after simplification)

\[(2.20) \quad h_{k-1}(x_1) = \frac{f_k(x_1)}{18k}.\]

Putting (2.20) into (2.14) and simplifying, we obtain (2.13). Therefore, \( h_k \) is a polynomial and the induction is complete. \( \square \)

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References


