Approach to chaos: Universal quantitative properties of one-dimensional maps

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Using the renormalization-group method, numerical results, and analytical arguments, we obtain the universal metric properties of one-dimensional iterated maps which exhibit period doublings. Such maps can be classified according to their behaviors around the extremum. For maps with the extremum (at $x=0$) of the symmetric form $f(x) \sim |x|^z b(x)$, with $z > 1$, where the function $b(x)$ modifies $|x|^z$ by less than any power of $|x|$, the quantitative universal properties of the period-doubling approach to chaos is described for asymptotically large n , by $\delta_n = (\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n) = \delta[1 + \epsilon(n)\tau \ln \alpha]$ and $\alpha_n = -d_n/d_{n+1} = \alpha[1 + \epsilon(n)\kappa \ln \alpha]$, where λ_n is the value of the parameter at the n th period-doubling bifurcation, d_n is the typical distance from the extremum at the n th bifurcation point, and $\epsilon(n) = [\partial \ln b(x)/\partial \ln |x|]_{|x| \sim \alpha^{-n}}$, where terms of order n^{-2} are neglected. Here δ , α , κ , and τ are all functions of z only; they obey the relations $d \ln \alpha / dz = \kappa \ln \alpha$ and $d \ln \delta / dz = \tau \ln \alpha$, where α and δ are the Feigenbaum constants. In particular, for $f(x) \approx |x|^z |\ln(x^2)|^p |\ln |\ln(x^2)||^q \dots$, where $z > 1$, then $d_n \approx \alpha^{-n} n^{\kappa p} (\ln n)^{\kappa q} \dots$ and $\lambda_\infty - \lambda_n \approx \delta^{-n} n^{\tau p} (\ln n)^{\tau q} \dots$, where λ_∞ is the critical value of the parameter beyond which is chaos. For $z=2$, $\alpha=2.50291$, $\delta=4.66920$, $\kappa=-0.44453$, and $\tau=0.36954$. For the sake of completeness, an analysis of maps with asymmetric extrema is also presented. In this case, α and δ are functions of z as well as the size of the asymmetry.

I. INTRODUCTION

Because of their relative simplicity and experimental interest, one-dimensional (1D) mappings of the real axis

$$f_\lambda: x \rightarrow x' = f(\lambda, x)$$

have been extensively studied.¹⁻³ A number of results have been obtained for some classes of functions. In particular, Feigenbaum² showed that if the function f_λ exhibits period doublings as the parameter λ is being increased, then f_λ has an infinite sequence of period-doubling bifurcations as λ approaches the critical value λ_c (beyond which f_λ exhibits chaotic behavior). Furthermore, the behavior of the bifurcations is quantitatively universal and depends only on the shape of f_λ near the extremum, $x=x_c$.

For example, if $\Delta f \approx |f_\lambda(x) - f_\lambda(x_c)| \approx |x - x_c|^z$, $z > 1$, both the bifurcation speed δ and the reduction scale factor α depend only on the power z . We can always make the shift so that the extremum is at the origin, i.e., $x_c=0$. For polynomial maps in general, the extremum has the quadratic behavior, i.e., $z=2$; the corresponding values for δ and α are $\delta=4.6692016$ and $\alpha=2.5029079$. It has been shown that the same universal properties are present in higher-dimensional dissipative maps as well as in experiments.

Feigenbaum's analysis is applicable to $\Delta f \approx |x|^z$ for all finite z , $z > 1$. In this paper we extend his analysis to the very general case where

$$\Delta f \approx |x|^z |\ln(x^2)|^p |\ln |\ln(x^2)||^q \dots, \quad (1.1)$$

where $p, q, \dots$ are real numbers. One-dimensional maps that exhibit infinite period-doubling bifurcation in general have their extrema of the form (1.1). [This form gives

classification of 1D symmetric maps that exhibit repeated period doublings; actually there is another class of 1D asymmetric maps,⁴ see (1.13). The analysis of the latter case is straightforward and a brief discussion is included in Sec. V.] Iterated maps with the property (1.1) reveal additional universal properties. For n asymptotically large, the value λ_n of the parameter λ at which n th bifurcation occurs behaves as

$$\lambda_\infty - \lambda_n \approx \delta^{-n} n^{\tau p} (\ln n)^{\tau q} \dots, \quad (1.2)$$

while the typical distance away from the extremum at the n th bifurcation point is

$$d_n \approx \alpha^{-n} n^{\kappa p} (\ln n)^{\kappa q} \dots, \quad (1.3)$$

where the function f_λ has a stable orbit with period 2^n when λ has a value $\lambda_n < \lambda \leq \lambda_{n+1}$; κ and τ are constants independent of $p, q, \dots$. (In the special case where $p=q=\dots=0$, Feigenbaum's result is recovered.) Like α and δ , they are functions of z only and they obey the following relations:

$$\frac{d \ln \alpha}{dz} = \kappa \ln \alpha, \quad (1.4)$$

$$\frac{d \ln \delta}{dz} = \tau \ln \alpha. \quad (1.5)$$

Using the renormalization-group method developed by Wilson and others,⁵ we can analytically derive the results (1.2) and (1.3) for the cases of p =positive integer and $q=0$. This analysis involves many interesting features. (For example, there appear $p+1$ marginal operators that take a Jordan's canonical form of order $p+1$.) This also enables us to analytically derive (1.4) and (1.5). κ and τ can now be obtained via essentially three ways:

- (1) numerical data from any function f_λ that exhibit (1.1),
- (2) renormalization-group analysis, and
- (3) Eqs. (1.4) and (1.5).

We obtain for $z=2$

$$\kappa = -0.444\,53 \cdots, \quad (1.6)$$

$$\tau = 0.369\,54 \cdots. \quad (1.7)$$

Although it is difficult to extend the renormalization-group method to $p \neq$ positive integer, the numerical data for $p \neq$ positive integer cases clearly indicate that the formulas (1.2) and (1.3) hold for any real number p , while $q=0$.

Assuming (1.2) and (1.3) to be correct for any real number p , an analytic-consistency argument shows that (1.2) and (1.3) are correct for large-integer values of q . Numerical data indicate that (1.2) and (1.3) are true also when q is a small integer, $q=1, 2, \dots$, as well as when q is not an integer. For $q \neq 0$ the approach to the asymptotic form (1.2) and (1.3) is extremely slow [corrections are of the order $(\ln n)^{-1}$]. However, our argument does indicate that the universal constants α , δ , τ , and κ essentially determine the infinite cascade of period doubling. From the details of the analysis and the numerical data, we conjecture that (1.3) is true for all real numbers $p, q, \dots$.

The renormalization-group results allow us to study the general symmetric case. In general, let us consider a function whose extremum at $x=0$ has the symmetric form

$$f \approx |x|^z b(x), \quad z > 1, \quad (1.8)$$

where the function $b(x)$ modifies the form $|x|^z$ by less than a power, i.e., as $x \rightarrow 0$,

$$\begin{aligned} |x|^\nu b(x) &\rightarrow 0, \\ |x|^{-\nu} b(x) &\rightarrow \infty, \end{aligned} \quad (1.9)$$

for any infinitesimal positive constant ν . Then, we argue, for asymptotically large n , that

$$\alpha_n = \frac{-d_n}{d_{n+1}} = \alpha[1 + \epsilon(n)\kappa \ln \alpha], \quad (1.10)$$

$$\delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta[1 + \epsilon(n)\tau \ln \alpha], \quad (1.11)$$

where

$$\epsilon(n) = \left. \frac{\partial \ln b(x)}{\partial \ln |x|} \right|_{|x| \approx \alpha^{-n}} \quad (1.12)$$

and terms of order n^{-2} are neglected. Readers who are only interested in this result can find the derivation in Sec. VB. The simple, intuitively clear argument used in the derivation in Sec. VB is justified by the careful, detailed renormalization-group analysis in Secs. III and IV. On the other hand, the derivation of (1.10)–(1.12) extends the renormalization-group results for $p =$ integer in (1.1)–(1.3) to all real values of $p, q, \dots$.

The constants α , δ , κ , and τ also determine the asymptotic band-bifurcation properties on the chaotic regime.³ It will be interesting to study the effect of noise on such

systems.

For the sake of completeness, we also discuss the case⁴ where the extremum is asymmetric, i.e.,

$$\Delta f \approx \begin{cases} a(1+\epsilon)|x|^z & \text{for } x > 0 \\ a(1-\epsilon)|x|^z & \text{for } x < 0, \end{cases} \quad (1.13)$$

where $\epsilon \neq 0$. In this case the universal equation for the universal function $g(x)$ is different from Feigenbaum's case, so that α and δ are functions of both z and the magnitude of the asymmetry ϵ .

Feigenbaum's analysis² is applicable to all functions with the extrema of the form $|x|^z$. As is clear from general arguments, α and δ have the following behaviors: as $z \rightarrow 1$, $\alpha \rightarrow \infty$ and $\delta \rightarrow 2$; as $z \rightarrow \infty$, $\alpha \rightarrow 1$ and $\delta \rightarrow \infty$. Since the period-doubling approach to chaos requires finite, well-defined α and δ , the extremum of the function can only be modified by behaviors slower than powers of x , if the period-doubling property is to be preserved. We argue that the only generalization of the Feigenbaum case is the introduction of logarithmic and/or asymmetric behaviors. This leads us to believe that our analysis is a complete analysis of the one-dimensional maps that exhibit the period-doubling approach to chaos. Note that all the functions that are considered in our analysis are at least once differentiable at the extrema.

The analysis of the universal properties of 1D maps certainly gives us a better understanding of the mathematical structures of iterated maps. It also demonstrates how the renormalization-group method⁵ can be used both as an analytical and a numerical tool. Although we do not know of any physical application of the type of maps discussed here, it is possible that they are relevant in certain situations. In quantum-field theory a massless scalar field x (e.g., a field responsible for spontaneous symmetry breaking) has a unique renormalizable self-interaction $f(x) \approx \lambda x^4$. Quantum effects due to its interactions with gauge fields and/or massless fermion fields and/or massless scalar fields will change it to $f(x) \approx x^4 \ln(x^2)$, which is mainly due to the presence of infrared singularities.⁶ Such a functional form plays a very important role in the study of the early universe in cosmology.

On the other hand, effective one-dimensional maps of systems with finite degrees of freedom sometimes show nonpolynomial extrema. For dissipative systems, one can always obtain one-dimensional maps via Poincaré section or via projection. Even if the original system has only polynomial-type couplings, nonpolynomial extremum can appear in the effective one-dimensional map obtained by these procedures. For example, in the Lorentz model⁷ there is a variable Z which measures the distortion of the vertical temperature profile; the one-dimensional map of M_n , the n th maximum value of Z , versus M_{n+1} , the value of the following maximum, displays a nonpolynomial extremum. One can easily envision the possibility of extrema with logarithmic modifications to the polynomial form in some nontrivial model.

If a logarithmic modification such as (1.1) is present, typically the transient behavior can be more persistent, and the values δ_n and α_n approach their respective universal values δ and α at a much slower rate. A comparison

TABLE I. Values of the superstable points $\tilde{\lambda}_n$ for the logistic equation (2.1) and the corresponding δ_n and α_n for the 2ⁿth iteration. Our calculations of $\tilde{\lambda}_n$ are performed with a ~ 30 significant-figure accuracy. Here, we display only the first 20 significant figures.

n	$\tilde{\lambda}_n$	δ_n	α_n
1	0.809 016 994 374 947 424 10		
2	0.874 640 424 831 925 380 00	4.708 943 013 540 50	2.627 369 964 258 67
3	0.888 660 215 692 206 216 34	4.680 770 998 010 70	2.539 984 368 426 16
4	0.891 666 844 964 067 128 49	4.662 959 611 114 10	2.505 384 991 479 15
5	0.892 310 882 909 277 584 45	4.668 403 925 918 40	2.503 820 081 261 51
6	0.892 448 823 437 486 155 13	4.668 953 740 967 62	2.503 050 932 063 43
7	0.892 478 366 355 587 128 71	4.669 157 181 328 84	2.502 944 520 100 44
8	0.892 484 693 558 326 371 95	4.669 191 002 485 10	2.502 914 970 191 95
9	0.892 486 048 652 016 233 31	4.669 199 470 547 73	2.502 909 486 756 76
10	0.892 486 338 871 617 145 22	4.669 201 134 601 04	2.502 908 208 857 41
11	0.892 486 401 027 769 609 53	4.669 201 509 513 55	2.502 907 947 985 45
12	0.892 486 414 339 714 124 93	4.669 201 587 522 39	2.502 907 890 532 47
13	0.892 486 417 190 724 992 09	4.669 201 604 512 19	2.502 907 878 423 46
14	0.892 486 417 801 324 213 63	4.669 201 608 115 94	2.502 907 875 805 89
15	0.892 486 417 932 095 868 55	4.669 201 608 892 07	2.502 907 875 248 28

of Tables I and II should illustrate this point. This means an accurate experimental measurement may reveal the presence or absence of logarithmic factors. From Table I we see, for $n \geq 4$, that the values of δ_{n-1} and α_{n-1} are essentially those of δ and α (difference $< 0.1\%$) while the values of $\delta_{n-1}(p=1)$ and $\alpha_{n-1}(p=1)$ in Table II deviate from the asymptotic values δ and α by a few percent. In experiments that produce power spectra the drop in power per period doubling will deviate from the value predicted.⁸ In fact, an accurate experiment that can reach the fifth or sixth period doubling may even provide a measurement of p , κ , and τ .

The renormalization-group (RG) method is a mathematical technique to treat problems which have scaling properties. It plays a crucial role in many areas of physics, e.g., quantum-field theory and critical phenomena. However, in these applications of the RG method, an in-depth knowledge of the relevant physics is necessary.

This prevents many nonphysicists from learning the RG method, which is simply a powerful mathematical tool. In this work the RG method plays a central role in the analysis. Readers with no physics background can appreciate the power and the working of the renormalization-group method itself.

This paper is organized as follows. In Sec. II the numerical data for $p \neq 0$ are presented. The data clearly suggest the existence of the universal constants κ and τ . In Sec. III a renormalization-group approach is presented for the case where $p=1$, $q=0$. This allows us to calculate κ and τ from the renormalization-group equations. Because of the length of this section, many of the details are relegated to the three appendixes. The generalization to p =positive integer is carried out in Sec. IV. This allows us to derive analytically the relations (1.4) and (1.5). In Sec. IV we also discuss the numerical results from the renormalization-group analysis. In Sec. V we discuss the

TABLE II. Values of the superstable points $\tilde{\lambda}_n$ for Eq. (2.14) and the corresponding δ_n and α_n .

n	$\tilde{\lambda}_n$	δ_n	α_n
1	0.508 654 307 287 357 597 82		
2	0.511 914 857 662 822 218 96	2.654 247 378 749 48	1.855 277 818 020 36
3	0.512 751 035 412 480 542 78	3.899 350 798 077 25	2.445 465 916 475 08
4	0.512 947 562 731 796 105 75	4.254 765 966 230 26	2.564 952 299 406 41
5	0.512 992 432 281 814 073 97	4.379 970 809 532 58	2.591 577 316 840 06
6	0.513 002 550 711 239 464 88	4.434 438 204 942 54	2.592 670 050 197 62
7	0.513 004 816 087 021 652 21	4.466 556 720 943 24	2.588 339 310 397 27
8	0.513 005 320 695 830 759 42	4.489 370 263 264 78	2.583 139 693 964 07
9	0.513 005 432 653 870 247 16	4.507 124 378 168 92	2.578 164 023 681 66
10	0.513 005 457 414 850 090 12	4.521 551 255 151 76	2.573 628 473 892 20
11	0.513 005 462 876 549 436 09	4.533 566 986 113 10	2.569 541 605 201 84
12	0.513 005 464 078 574 878 54	4.543 746 873 486 86	2.565 863 594 391 83
13	0.513 005 464 342 611 934 00	4.552 487 681 538 59	2.562 546 936 601 70
14	0.513 005 464 400 513 819 27	4.560 077 003 058 58	2.559 546 830 069 10
15	0.513 005 464 413 192 888 97	4.566 729 788 697 50	2.556 823 611 379 37
16	0.513 005 464 415 965 718 67	4.572 610 305 479 18	2.554 342 935 528 68

generalization to the case (1.1). The evidence for the form (1.2) and (1.3) is partly numerical and partly analytic. Our argument clearly indicates the behaviors of the most general symmetric case (1.8)–(1.12). In this section the theory of the asymmetric case (1.13) is also briefly discussed. Section VI contains the conclusion as well as some remarks. Appendix A discusses the effect of a change in the choice of the marginal operator $s(x)$. Appendix B contains the proof that κ and τ are invariant under a change of the operator $s(x)$. A number of relations among the coefficients of the set of renormalization-group equations are derived. Appendix C gives some properties of $s(x)$ in relation to κ . Here, the functional relation (1.4) is derived using a different approach than that in Sec. IV.

II. NUMERICAL DATA AND INTERPRETATION

To be specific, let us consider the following one-dimensional iterative map (the logistic equation)

$$x_{n+1} = f(\lambda, x_n) = 4\lambda x_n(1 - x_n), \quad (2.1)$$

where x_n is a real number, $x_n \in [0, 1]$. This map is well studied.^{1,9} Our review allows us to introduce the notations and set up the stage for generalization. Depending on the value of λ this map may or may not have a stable orbit with a definite period. It is easy to show that if $\frac{1}{4} < \lambda \leq \frac{3}{4}$ there is a stable (i.e., attractive) fixed point at $x^* = 1 - 1/4\lambda$, i.e.,

$$x^* = f(\lambda, x^*), \quad (2.2)$$

such that repeated iterations of (2.1) will bring x_n towards x^* . This is a stable fixed point. As λ increases above $\frac{3}{4}$ this fixed point becomes unstable while a pair of new stable points, x_1 and x'_1 , appears so that

FIG. 1. Attractor, i.e., the stable orbit with period 2^n , vs the bifurcation parameter λ for a typical map that exhibits a cascade of period doublings. The figure is schematic. $\tilde{\lambda}_i$ are the superstable points. Δ_i is the slope at the point $\tilde{\lambda}_i$. The extremum is taken to be at $x_c = \frac{1}{2}$. λ_c is the critical point.

$$x_1 = f(\lambda, x'_1),$$

$$\begin{aligned} x'_1 &= f(\lambda, x_1) = f(\lambda, f(\lambda, x'_1)) = f \circ f(\lambda, x'_1) \\ &= f^2(\lambda, x'_1). \end{aligned} \quad (2.3)$$

They are stable in the sense that points in their neighborhood will approach them under repeated iterations. They form a stable orbit with period 2. λ_1 is the value of λ at which the bifurcation from period 1 to period 2 occurs. In general λ_n is the value of λ at which the n th bifurcation occurs.

In Fig. 1 the stable orbit with period 2^n , $\{x_n\} = \{x_n^{(i)}, i = 1, 2, \dots, 2^n\}$, is shown as a function of λ . This figure is schematic and is generic for all 1D iterated maps which exhibit cascade of period-doubling bifurcation and have $x_c = \frac{1}{2}$. Let us define

$$\delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}, \quad (2.4)$$

$$\alpha_n = -\frac{d_n}{d_{n+1}} \quad (2.5)$$

then for large n , $\delta_n \rightarrow \delta$ and $\alpha_n \rightarrow \alpha$, where α and δ are the Feigenbaum constants. It is clear that for some value of $\lambda (\equiv \tilde{\lambda}_n)$, x_c is just on the stable orbit of period 2^n , where

FIG. 2. Values of κ from Table III, which are extracted from the numerical data of α_n in Table II for the function (2.16). Curve 1 is the one-parameter fit, while curve 2 includes a correction term of the form A/n . Curve 3 includes correction terms of the form $(A/n) + (B/n^2)$ and curve 4 includes terms of the form $(A/n) + (B/n^2) + (C/n^3)$. Curves 1, 2, 3, and 4 correspond to $\kappa_1, \kappa_2, \kappa_3$, and κ_4 in Table III. The inputs are discrete points at integer values of n ; the curve is to guide the reader.

$\lambda_n < \tilde{\lambda}_n < \lambda_{n+1}$. For large enough n , we can use $\tilde{\lambda}_n$ for an alternative, equivalent (but not equal) definition for δ_n ,

$$\delta_n = \frac{\tilde{\lambda}_n - \tilde{\lambda}_{n-1}}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_n}. \quad (2.6)$$

$\tilde{\lambda}_n$ are known as the superstable points. We note that the distance d_n vanishes at the superstable points. So, instead of Eq. (2.5), an equivalent scale factor can be defined at $\tilde{\lambda}_n$,

$$\alpha_n = -\delta_n \frac{\Delta_n}{\Delta_{n+1}}, \quad (2.7)$$

where Δ_n is the slope at $\tilde{\lambda}_n$. In Table I the $\tilde{\lambda}_n$, δ_n , and α_n are shown for n up to $n=15$ for the logistic equation (2.1). It is clear that α_n and δ_n rapidly approach their respective universal values α and δ . This implies

$$\Delta \tilde{\lambda}_n \approx \tilde{\lambda}_n - \tilde{\lambda}_{n-1} \approx \delta^{-n}, \quad (2.8)$$

$$d_n \approx \alpha^{-n}. \quad (2.9)$$

This universal quantitative behavior is present because the behavior of the iterated map for large n is dominated by its property around the extremum. Hence, if $f_\lambda(x)$ is a general polynomial, its extremum will be quadratic (i.e., $z=2$). Therefore its period-doubling behavior will be given by (2.8) and (2.9) for sufficiently large n . In practice (2.8) and (2.9) are good approximations when $n \geq 3$ or 4. If we change (2.1) to

$$x_{n+1} = \lambda(1 - c |x_n - \frac{1}{2}|^z), \quad z > 1 \quad (2.10)$$

then the period-doubling-bifurcation behavior will again obey (2.8) and (2.9). But the values of α and δ will be different, i.e., $\alpha = \alpha(z)$ and $\delta = \delta(z)$. For $z \rightarrow 1$, $\alpha \rightarrow \infty$ and $\delta \rightarrow 2$. For $z \rightarrow \infty$, $\alpha \rightarrow 1$ and $\delta \rightarrow \infty$. The period-doubling property is present for an appropriate range of values of the coefficient c . It is easy to convince oneself (via numerical data, for example) that $\alpha(z)$ and $\delta(z)$ are differentiable functions of z , $z \in (1, \infty)$.

Now let us consider the following map, $z > 1$:

$$x_{n+1} = \lambda(1 - c |x_n - \frac{1}{2}|^z |\ln(x_n - \frac{1}{2})^2|^p), \quad (2.11)$$

where p is a real number. Here, the extremum at $x_c = \frac{1}{2}$ is modified by logarithm. The effect of the logarithmic modification is less than that of a power modification, no matter how small the change in power is. Therefore, we expect $\alpha_n, \delta_n \rightarrow \alpha(z)$ and $\delta(z)$. However, the speed of approach may be modified. For $p \neq 0$ we expect $\Delta \tilde{\lambda}_n$ in Eq. (2.8) and d_n in Eq. (2.9) to be functions of p as well as α , δ , and n . As we shall see, the numerical data indicate that, for any p and z , we have, for large n ,

$$\tilde{\lambda}_\infty - \tilde{\lambda}_n \approx \delta^{-n} n^{p\tau}, \quad (2.12)$$

$$d_n \approx \alpha^{-n} n^{p\kappa}, \quad (2.13)$$

where α , δ , κ , and τ are functions of only z .

In Table II the numerical data for $z=2$, $p=1$ are given. The particular function used is

$$f(\lambda, x) = \lambda \{ 1 + 7(2x - 1)^2 \ln[(2x - 1)^2] \}. \quad (2.14)$$

It is useful to compare the δ_n and α_n in Tables I and II. In Table II the approach of $\delta_n, \alpha_n \rightarrow \delta, \alpha$ is much slower and less apparent (if at all). To check the forms (2.12) and (2.13) we note that (2.12) and (2.13) imply

$$\begin{aligned} \delta_n &= \delta \left[1 - \frac{p\tau}{n} + O\left(\frac{1}{n^2}\right) \right], \\ \alpha_n &= \alpha \left[1 - \frac{p\kappa}{n} + O\left(\frac{1}{n^2}\right) \right]. \end{aligned} \quad (2.15)$$

In Fig. 2, κ versus n is shown. Since (2.13) and (2.15) are expected to be correct only for large n , we have to consider

$$\frac{\alpha_n}{\alpha} = 1 - \frac{p\kappa}{n} + \frac{A}{n^2} + \frac{B}{n^3} + \frac{C}{n^4}, \quad (2.16)$$

where the constants κ , A , B , and C are to be determined via data fitting.

Curve 1 in Fig. 2 is obtained with the constants A, B, C set to zero. Curve 2 has $B=C=0$, while κ, A are determined by comparing α_{n-1} and α_n . Curve 3 has $C=0$ while κ, A , and B are determined by $\alpha_{n-1}, \alpha_n, \alpha_{n+1}$. Curve 4 gives κ where κ, A, B , and C are determined by $\alpha_{n-2}, \alpha_{n-1}, \alpha_n$, and α_{n+1} . In Fig. 3, τ versus n is shown, where a similar fitting has been performed. In Tables III and IV are the numerical κ and τ obtained from Table II. If we assume the constants κ and τ exist, then $\kappa \approx -0.44$ and $\tau \approx 0.37$. Next, let us consider some functions with the same z , $z=2$, but different values of p . To be specific, we consider

$$f(x) = \lambda \{ 1 - (2x - 1)^2 [c |\ln(2x - 1)|^p + d] \},$$

where the $(c, d, p) = (\frac{2}{5}, 1, 2), (7, 0, 1), (1, 0, \frac{1}{2}), (1, 0, -\frac{1}{20}), (1, 0, -\frac{1}{10}), (1, 0, -\frac{1}{5}), (\frac{3}{2}, 0, -1)$, and $(27, 0, -2)$. In each case we extract the κ and τ using formulas similar to Eq.

FIG. 3. Values of τ from Table IV, which are extracted from the numerical data of δ_n in Table II for the function (2.16). Curves 1, 2, 3, and 4 correspond to τ_1, τ_2, τ_3 , and τ_4 in Table IV.

TABLE III. Value of κ extracted from the α_n values in Table II. κ_1 is given by $\alpha_n/\alpha = 1 - (\kappa_1/n)$; κ_2 is given by the fit $\alpha_n/\alpha = 1 - (\kappa_2/n) + (A/n^2)$ and its value can be obtained by comparing α_n and α_{n-1} . Similarly, $\alpha_n/\alpha = 1 - (\kappa_3/n) + (A/n^2) + (B/n^3)$ and $\alpha_n/\alpha = 1 - (\kappa_4/n) + (A/n^2) + (B/n^3) + (C/n^4)$.

n	κ_1	κ_2	κ_3	κ_4
4	-0.097 946 9	-0.600 745 0	-0.295 259 5	-0.350 250 6
5	-0.174 067 7	-0.478 550 8	-0.237 273 5	-0.179 287 5
6	-0.211 410 6	-0.398 125 0	-0.321 174 9	-0.433 043 4
7	-0.234 943 2	-0.376 139 3	-0.370 595 5	-0.452 963 2
8	-0.252 419 5	-0.374 753 3	-0.399 344 1	-0.456 841 3
9	-0.266 619 3	-0.380 217 9	-0.413 060 3	-0.445 064 6
10	-0.278 636 0	-0.386 786 4	-0.420 067 2	-0.438 752 3
11	-0.289 018 0	-0.392 837 5	-0.424 069 6	-0.436 076 7
12	-0.298 103 4	-0.398 042 8	-0.426 732 0	-0.435 606 8
13	-0.306 130 5	-0.402 456 5	-0.428 715 5	-0.435 988 5
14	-0.313 278 9	-0.406 207 8	-0.430 297 0	-0.436 623 1
15	-0.319 688 3	-0.409 419 7	-0.431 604 8	-0.437 271 7

(2.16). The results are shown in Figs. 4 and 5. Curve 4 in Figs. 2 and 3 are repeated in Figs. 4 and 5. The numerical data do provide good evidence that the constants κ and τ exist for both positive and negative real p , with $\kappa \approx -0.445$ and $\tau \approx 0.370$ where errors are of the order of a few percent.

It is very difficult to improve the accuracy of κ and τ via straight numerical computation. This is because the computation time increases exponentially with n while the accuracy increases only like inverse powers of n . For large $|p|$ the numerical determinations of κ and τ become worse.

In the rest of this paper we shall use the renormalization-group method

- (1) to verify the form (2.12) and (2.13) for all positive integer p ,
- (2) to determine κ and τ to better precisions,
- (3) to elucidate the functional relations among α , δ , κ , and τ , and
- (4) to extend the result to the general form

$$f(x) - f(x_c) \approx |x - x_c|^z |\ln |x - x_c||^p \times |\ln |\ln |x - x_c|||^q \dots$$

by a combination of analytical and numerical arguments.

Feigenbaum's analysis of the iterated maps uses the renormalization-group method. In fact, his work provides the starting point of our investigation. After the renormalization-group analysis, we derive the universal quantitative properties for the general case.

III. THE $p = 1$ CASE

In this section we consider only the $p = 1$ case. A brief review of Feigenbaum's renormalization-group analysis² is followed by its generalization to the case relevant to the $p = 1$ case. Equation (2.15) [hence (2.12) and (2.13)] shall be derived analytically for the $p = 1$ case. Some of the details are relegated to the three appendices.

A. Review

For some general function $f(\lambda, x)$, its m th iteration is given by

$$f^m(\lambda, x) = f(\lambda, f(\lambda, f(\lambda, \dots))) . \quad (3.1)$$

Without loss of generality we can set the extremum at the origin, i.e., $x_c = 0$. Let

TABLE IV. Value of τ extracted from the δ_n values in Table II. τ_1 , τ_2 , τ_3 , and τ_4 are obtained in a way similar to that for κ in Table III.

n	τ_1	τ_2	τ_3	τ_4
4	0.371 793 6	-0.134 428 4	0.480 337 5	0.554 663 5
5	0.319 730 4	0.111 478 0	0.552 506 5	0.624 675 4
6	0.309 523 3	0.258 487 5	0.463 306 5	0.344 373 3
7	0.310 592 4	0.317 007 2	0.406 972 3	0.313 082 0
8	0.314 205 7	0.339 498 5	0.377 701 5	0.319 159 7
9	0.317 959 3	0.347 988 0	0.366 388 0	0.339 989 9
10	0.321 330 1	0.351 668 0	0.362 679 3	0.352 789 6
11	0.324 270 1	0.353 670 1	0.361 873 9	0.359 457 7
12	0.326 834 1	0.355 037 4	0.362 017 4	0.362 495 6
13	0.329 086 2	0.356 111 2	0.362 421 6	0.363 903 9
14	0.331 080 9	0.357 012 7	0.362 863 2	0.364 629 3
15	0.332 861 7	0.357 792 8	0.363 278 4	0.365 077 8

FIG. 4. Value of τ extracted from functions with different p values. The p value for each case is displayed next to the curve; all the curves correspond to τ_4 .

FIG. 5. Value of κ extracted from functions with different p values. The p value for each case is displayed next to the curve. All the curves correspond to κ_4 .

$$\tilde{g}_n(x) \equiv (-1)^n \beta_n f^{2^n} \left[\lambda_\infty, \frac{x}{\beta_n} \right], \quad (3.2)$$

where λ_∞ is the λ that has a stable orbit with an infinite period. This is the critical value beyond which is chaos. For the sake of simplicity we assume $\tilde{g}_n(x)$ to be symmetric, i.e., $\tilde{g}_n(x) = \tilde{g}_n(-x)$. Replacing x by $\beta_n x$ we have

$$\frac{1}{\beta_n} \tilde{g}_n(\beta_n x) = (-1)^n f^{2^n}(\lambda_\infty, x). \quad (3.3)$$

Then, using (3.3), we obtain

$$\frac{-1}{\beta_{n+1}} \tilde{g}_{n+1}(\beta_{n+1} x) = \frac{1}{\beta_n} \tilde{g}_n(\tilde{g}_n(\beta_n x)) \quad (3.4)$$

or, replacing x by x/β_{n+1} ,

$$I[\alpha_n, \tilde{g}_n] \equiv -\alpha_n \tilde{g}_n(\tilde{g}_n(x/\alpha_n)) = \tilde{g}_{n+1}(x), \quad (3.5)$$

where $\alpha_n = \beta_{n+1}/\beta_n$. If the limit $n \rightarrow \infty$ exists for (3.5) then $\tilde{g}_n(x) \rightarrow g(x)$, $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$, and

$$I[\alpha, g] = -\alpha g(g(x/\alpha)) = g(x), \quad (3.6)$$

where $g(x)$ is the universal function. Since the normalization is not fixed, we can define

$$g(0) = 1 \quad (3.7)$$

which gives $g(1) = -1/\alpha$. Functionally, $g(x)$ can be written as

$$g_z(x) = 1 + g_1 |x|^z + \dots, \quad (3.8)$$

where the coefficients as well as α can be determined numerically by using Eq. (3.6). From (3.6) and (3.8) it is easy to obtain

$$g'_z(1) = -\alpha^{z-1}. \quad (3.9)$$

For large n we write

$$\tilde{g}_n(x) = g(x) + \sum_i c_n^{(i)} E_i(x), \quad (3.10)$$

where $E_i(x)$ are a set of functions to be determined. For large n the coefficients $c_n^{(i)}$ are expected to be small. Substituting Eq. (3.10) into Eq. (3.5) and using Eq. (3.6), we have

$$\sum_i c_{n+1}^{(i)} E_i(x) = \sum_i c_n^{(i)} L[\alpha_n, E_i] + \dots, \quad (3.11)$$

where the operator $L[\alpha_n, E_i]$, the linearized operator of $I[\alpha_n, \tilde{g}_n]$, is defined by

$$L[\alpha_n, E_i] \equiv -\alpha_n \left[g'_z \left[g_z \left[\frac{x}{\alpha_n} \right] \right] E_i \left[\frac{x}{\alpha_n} \right] + E_i \left[g_z \left[\frac{x}{\alpha_n} \right] \right] \right] \quad (3.12)$$

and higher-order terms are neglected, and

$$g'_z(x) = \frac{dg_z(x)}{dx}.$$

Since L is a linear operator, its eigenvalues and eigenvectors

tors can be determined (numerically or analytically). In the case without logarithms, it is found that there is one eigenvector with eigenvalue larger than one; its value is δ ($\delta > 2$):

$$L[\alpha, h] = \delta h(x). \quad (3.13)$$

In the renormalization-group language, $h(x)$ is a relevant operator. There is also an eigenvector with eigenvalue 1, i.e.,

$$L[\alpha, t] = t(x). \quad (3.14)$$

Eigenvectors with eigenvalue 1 are known as marginal operators. In this case it is easy to show that

$$t(x) = g(x) - xg'(x). \quad (3.15)$$

Because of Eq. (3.15), $t(x)$ is also known as a redundant operator. To see its meaning, let us consider $\mu g(x/\mu)$, which clearly satisfies (3.6). For $\mu = 1 + \epsilon$, ϵ small, we have

$$R[\epsilon, g(x)] \equiv (1 + \epsilon)g(x/(1 + \epsilon)) \\ = g(x) + \epsilon t(x) + \frac{\epsilon^2 x^2 g''(x)}{2} + O(\epsilon^3). \quad (3.16)$$

Therefore $t(x)$ is the rescaling operator. In the absence of logarithms, $\alpha_n \approx \alpha$, and (3.11) can be written as

$$u_{n+1}h(x) + w_{n+1}t(x) + \sum_j c_{n+1}^{(j)}k_j(x) \\ = \delta u_n h(x) + w_{n+1}t(x) + \sum_j e^{(j)}c_n^{(j)}k_j(x) + \dots, \quad (3.17)$$

where k_j is an eigenvector with eigenvalue $e^{(j)}$, where $e^{(j)} < 1$. $k_j(x)$ is known as an irrelevant operator since its coefficient $c_n^{(j)} \approx (e^{(j)})^n$ will in general decrease geometrically and become negligible rapidly. Equation (3.17) is valid because $h(x)$, $t(x)$, and $k_j(x)$ form a complete set of eigenfunctions. To leading order, we have, from (3.17),

$$u_{n+1} = \delta u_n + \dots, \quad (3.18)$$

$$w_{n+1} = w_n + \dots$$

If we choose a normalization such that for some n , say n_0 , such that $w_{n_0} = 0$, then $w_n = 0$ for $n \geq n_0$, and, for large n , $u_n \simeq c\delta^n$, for some constant c . This means the approach to chaos is geometric. δ in Eq. (3.18) is the Feigenbaum constant.

B. The $p = 1$ operator

In the case where $\tilde{g}_n(x)$ is expected to have a logarithmic factor of the form $|x|^2 \ln(x^2)$, we expect

$$\tilde{g}_n(x) = g(x) + u_n h(x) + w_n t(x) + v_n s(x) + \dots, \quad (3.19)$$

where $s(x) \approx |x|^2 \ln(x^2) + \dots$. For $x \rightarrow 0$, this term should dominate and, using (3.9),

$$L[\alpha, s] \approx -\alpha[g'(1)s(x/\alpha) + \dots] \\ \approx |x|^2 \ln(x^2) + \dots,$$

indicating that the corresponding eigenvalue is 1 and hence $s(x)$ is a marginal operator. However, $s(x)$ is not an eigenvector, as we now demonstrate. To present the z dependence more explicitly, let us write (3.6) as

$$I[\alpha, g] = -\alpha_z g_z(g_z(x/\alpha_z)) = g_z(x). \quad (3.20)$$

Similarly, for small ϵ , we have

$$-\alpha_{z+\epsilon} g_{z+\epsilon}(g_{z+\epsilon}(x/\alpha_{z+\epsilon})) = g_{z+\epsilon}(x), \quad (3.21)$$

where

$$g_{z+\epsilon}(x) = g_z(x) + \epsilon \frac{\partial g_z}{\partial z}(x) + \dots, \quad (3.22)$$

$$\alpha_{z+\epsilon} = \alpha_z + \epsilon \frac{\partial \alpha_z}{\partial z} + \dots. \quad (3.23)$$

From (3.8) we have

$$\tilde{s}(x) \equiv \frac{\partial g_z}{\partial z}(x) \\ = \left[\frac{\partial g_1}{\partial z} \right] |x|^z + \frac{g_1}{2} |x|^2 \ln(x^2) + \dots. \quad (3.24)$$

Substituting (3.22)–(3.24) into (3.21) and using (3.20), we obtain, for terms with the leading power in ϵ ,

$$-\alpha \left[g'_z \left[g_z \left[\frac{x}{\alpha} \right] \right] \tilde{s}(x) + \tilde{s} \left[g_z \left[\frac{x}{\alpha} \right] \right] \right] \\ - \frac{1}{\alpha^2} \left[\frac{\partial \alpha_z}{\partial z} \right] g'_z \left[g_z \left[\frac{x}{\alpha} \right] \right] x g'_z \left[\frac{x}{\alpha} \right] \\ + \frac{1}{\alpha} \left[\frac{\partial \alpha_z}{\partial z} \right] g_z \left[g_z \left[\frac{x}{\alpha} \right] \right] \right] = \tilde{s}(x).$$

Using (3.15) we obtain

$$L[\alpha, s] = s(x) + t(x), \quad (3.25)$$

where

$$s(x) = -\alpha_z \left[\frac{\partial \alpha_z}{\partial z} \right]^{-1} \tilde{s}(x) \\ = -\alpha_z \left[\frac{\partial \alpha_z}{\partial z} \right]^{-1} \frac{\partial g_z(x)}{\partial z}. \quad (3.26)$$

Equations (3.14) and (3.25) show that the operator L is not symmetric, so that it takes the Jordan canonical form. $s(x)$ is the new marginal operator for $p = 1$. We observe that if $s(x)$ is a solution of (3.25), then so is $s(x) - rt(x)$ for any constant r . We shall come back to this point later.

C. The renormalization-group equation

Using (3.5) and (3.19) and Taylor expansion, it is straightforward to obtain, for n large and small coefficients u_n , v_n , w_n ,

$$g(x) + u_{n+1}h(x) + w_{n+1}t(x) + v_{n+1}s(x) + \cdots = I[\alpha_n, g_n] \quad (3.27)$$

$$= -\alpha_n \tilde{g}_n(\tilde{g}_n(x/\alpha_n)) \quad (3.28)$$

$$= -\alpha_n g(g(x/\alpha_n)) + u_n L[\alpha_n, h] + w_n L[\alpha_n, t] + v_n L[\alpha_n, s] + \frac{u_n^2}{2} Q[\alpha_n, h, h] \\ + u_n w_n Q[\alpha_n, h, t] + u_n v_n Q[\alpha_n, h, s] + \frac{w_n^2}{2} Q[\alpha_n, t, t] + w_n v_n Q[\alpha_n, t, s] + \frac{v_n^2}{2} Q[\alpha_n, s, s] + \cdots, \quad (3.29)$$

where terms from the irrelevant operators are neglected on both sides. The terms that are cubic in the coefficients u_n, v_n, w_n are ignored in the right-hand side. The operator L is defined in (3.12) and the operator Q is defined as follows:

$$Q[\alpha_n, t, h] = Q[\alpha_n, h, t] = -\alpha_n \left[g'' \left[g \left[\frac{x}{\alpha_n} \right] \right] h \left[\frac{x}{\alpha_n} \right] t \left[\frac{x}{\alpha_n} \right] \right. \\ \left. + h' \left[g \left[\frac{x}{\alpha_n} \right] \right] t \left[\frac{x}{\alpha_n} \right] + h \left[\frac{x}{\alpha_n} \right] t' \left[g \left[\frac{x}{\alpha_n} \right] \right] \right] \quad (3.30)$$

and similarly for the other Q operators. Let us consider the various terms on the right-hand side of Eq. (3.29). By using (3.6) the first term can be written as

$$-\alpha_n g(g(x/\alpha_n)) = (\alpha_n/\alpha) g(\alpha x/\alpha_n) \\ = R[\epsilon_n, g(x)] \\ = g(x) + \epsilon_n t(x) \\ + \epsilon_n^2 \frac{x^2}{2} g''(x) + O(\epsilon_n^3), \quad (3.31)$$

where (3.16) is used. $\epsilon_n \equiv -1 + \alpha_n/\alpha$ is taken to be small for large n .

Using (3.13) on the second term on the right-hand side (rhs) of (3.27), we have

$$L[\alpha_n, h] = \delta(1 + \epsilon_n) h(x/(1 + \epsilon_n)) \\ = \delta R[\epsilon_n, h(x)] \\ = \delta \{ h(x) + \epsilon_n [h(x) - xh'(x)] \} \quad (3.32)$$

and similarly for the next two terms. The goal is to express the terms in (3.27) in terms of $g(x)$, $h(x)$, $t(x)$, $s(x)$, and the irrelevant operators. $h(x)$, $t(x)$, $s(x)$, and the irrelevant operators should form a complete basis so that we can write

$$\frac{x^2 g''(x)}{2} = d_1 h(x) + e_1 t(x) + f_1 s(x) + \cdots, \\ h(x) - xh'(x) = d_2 h(x) + e_2 t(x) + f_2 s(x) + \cdots, \\ t(x) - xt'(x) = d_3 h(x) + e_3 t(x) + f_3 s(x) + \cdots, \\ s(x) - xs'(x) = d_4 h(x) + e_4 t(x) + f_4 s(x) + \cdots, \quad (3.33)$$

where d_i , e_i , and f_i are coefficients to be determined. The irrelevant operators are not written out explicitly. If we keep terms only up to second power in u_n, v_n, w_n and ϵ_n in Eq. (3.27), then $Q[\alpha_n, h, h] \simeq Q[\alpha, h, h]$, $Q[\alpha_n, h, t] = Q[\alpha, h, t]$, etc., so that

$$Q[\alpha, h, h] = a_1 h(x) + b_1 t(x) + c_1 s(x) + \cdots, \\ Q[\alpha, h, t] = a_2 h(x) + b_2 t(x) + c_2 s(x) + \cdots, \\ Q[\alpha, t, t] = a_3 h(x) + b_3 t(x) + c_3 s(x) + \cdots, \\ Q[\alpha, h, s] = a_4 h(x) + b_4 t(x) + c_4 s(x) + \cdots, \\ Q[\alpha, s, t] = a_5 h(x) + b_5 t(x) + c_5 s(x) + \cdots, \\ Q[\alpha, s, s] = a_6 h(x) + b_6 t(x) + c_6 s(x) + \cdots, \quad (3.34)$$

where a_i , b_i , and c_i are coefficients to be determined. We observe that some of the coefficients in (3.34) must be trivially zero. $g(x)$, $h(x)$, $t(x)$, and their derivatives are simple polynomials in x (and/or fractional powers of x). Hence, $f_1 = f_2 = f_3 = c_1 = c_2 = c_3 = 0$. (Other properties of the coefficients shall be discussed later.) Now we can equate the coefficients of $h(x)$, $t(x)$, and $s(x)$ [in Eqs. (3.27)–(3.29)] to obtain these coupled equations,

$$u_{n+1} = \delta u_n + \frac{a_1}{2} u_n^2 + a_2 u_n w_n + \frac{a_3}{2} w_n^2 \\ + a_4 u_n v_n + a_5 w_n v_n + \frac{a_6}{2} v_n^2 + \delta d_2 u_n \epsilon_n \\ + d_3 \epsilon_n w_n + (d_3 + d_4) \epsilon_n v_n + \cdots, \quad (3.35)$$

$$w_{n+1} = w_n + v_n + \epsilon_n + \frac{b_1}{2} u_n^2 + b_2 u_n w_n + \frac{b_3}{2} w_n^2 \\ + b_4 u_n v_n + b_5 w_n v_n + \frac{b_6}{2} v_n^2 + e_2 \delta u_n \epsilon_n + e_3 \epsilon_n w_n \\ + (e_3 + e_4) \epsilon_n v_n + e_1 \epsilon_n^2 + \cdots, \quad (3.36)$$

$$v_{n+1} = v_n + c_4 u_n v_n + c_5 w_n v_n + \frac{c_6}{2} v_n^2 + f_4 \epsilon_n v_n + \cdots, \quad (3.37)$$

where the v_n term appearing in (3.36) is due to (3.25).

These three coupled equations form the set of renormalization-group equations. To simplify this set of equations, we now make a number of observations.

(1) $t(x)$ is a simple function of $g(x)$, as given by (3.15); also, $t'(x) = -xg''(x)$. Hence, from (3.33), we obtain

$$\begin{aligned} d_3 &= 2d_1, \\ e_3 &= 2e_1 + 1. \end{aligned} \quad (3.38)$$

(2) Both w_n and ϵ_n are related to rescaling. Hence for each n , we can choose ϵ_n such that the rhs of (3.36) vanishes:

$$\epsilon_n = -v_n + \dots$$

Then w_n can be set to zero for all n ,

$$w_n = 0.$$

This simplifies the other two equations as well.

(3) u_n is taken to be much smaller than v_n . This is necessary since further iteration will increase u_n geometrically (i.e., $u_{n+m} \sim \delta^m u_n$ where $\delta > 1$), while the renormalization-group equations assume a perturbative expansion in u_n, v_n, w_n , etc. Clearly, the validity of the renormalization-group equations break down when $u_{n+m} \sim 1$ for some m (see Ref. 5 for a detailed argument). Hence, to the approximation we need, (3.35)–(3.37) reduce to a much simpler set of equations,

$$u_{n+1} = \delta u_n + (a_4 - d_2 \delta) u_n v_n, \quad (3.39)$$

$$v_{n+1} = v_n + \left[\frac{c_6}{2} - f_4 \right] v_n^2, \quad (3.40)$$

$$\alpha_n = \alpha(1 + \epsilon_n) = \alpha(1 - v_n), \quad (3.41)$$

where (3.41) follows from the definition of ϵ_n .

(4) We can now solve (3.40) to obtain, for large n ,

$$-\epsilon_n \approx v_n \approx \frac{-1}{\left[\frac{c_6}{2} - f_4 \right] n}. \quad (3.42)$$

Equations (3.40)–(3.42) verify Eq. (2.15) for the $p=1$ case. Comparing (3.41) and (3.42) with (2.15) for $p=1$, we obtain

$$\kappa = - \left[\frac{c_6}{2} - f_4 \right]^{-1}. \quad (3.43)$$

Substituting (3.42) into (3.39), we obtain

$$u_{n+1} = \delta \left[1 + \frac{(a_4 - d_2 \delta) \kappa}{n \delta} \right] u_n.$$

Comparing this with (2.15) and (3.18), we obtain

$$\tau = - \frac{\kappa}{\delta} (a_4 - d_2 \delta). \quad (3.44)$$

Hence we have shown analytically the asymptotic form (2.15) for the $p=1$ case since κ and τ are independent of n . Equations (2.12) and (2.13) immediately follow from (2.15). The coefficients c_6, f_4, a_4 , and d_2 can be determined numerically. This will be done in Sec. IV. Equations

(3.43) and (3.44) provide a renormalization-group method calculation of the constants κ and τ .

(5) There is one very important technical point that we have ignored so far. As mentioned earlier, if $s(x)$ is a solution of (3.25), then so is $s(x) - rt(x)$ for any constant r . It is clear from (3.33) and (3.34) that a change of $s(x)$ will cause corresponding changes in some of the coefficients. This is discussed in Appendix A. In particular, under $s(x) \rightarrow s(x) - rt(x)$, $a_4 \rightarrow a_4 - ra_2$ and $c_6 \rightarrow c_6 - 2rc_5$ while f_4 and d_2 remain invariant, so that Eq. (3.43) becomes

$$\kappa = - \left[\frac{c_6}{2} - rc_5 - f_4 \right]^{-1}$$

and Eq. (3.44) becomes

$$\tau = - \frac{\kappa}{\delta} (a_4 - ra_2 - d_2 \delta).$$

This will imply that κ and τ are independent of r if and only if a_2 and c_5 are exactly zero for all choices of r . This turns out to be precisely the case. The proof follows from the properties of the operators I and R and is presented in Appendix B. Here let us summarize the results. Independent of r , we obtain

$$\begin{aligned} a_2 &= b_3 = c_5 = 0, \\ a_3 &= (1 - \delta) d_3, \\ a_5 &= d_3 + (1 - \delta) d_4, \\ b_2 &= (1 - \delta) e_2, \\ b_5 &= 2(e_1 + 1), \end{aligned} \quad (3.45)$$

where we also have used the fact that

$$f_4 = 1 - z \quad (3.46)$$

which follows easily from an examination of (3.33) and Appendix C.

(6) It is clear, to calculate the constants κ and τ , that we must first determine the functions $g(x)$, $h(x)$, $t(x)$, and $s(x)$ numerically; then we evaluate the coefficients a_4 , c_6 , and d_2 . Here we want to show that c_6 and hence κ can be calculated in terms of the coefficients in the eigenfunction $s(x)$ and the universal function $g(x)$. In Appendix C we show that

$$\kappa^{-1} = - \left[\frac{c_6}{2} + z - 1 \right] = - \frac{2\beta_1 \ln \alpha}{g_1}, \quad (3.47)$$

where β_1 is the coefficient of $|x| \ln x^2$ in $s(x)$, i.e.,

$$s(x) = \beta_1 |x| \ln(x^2) + \dots, \quad (3.48)$$

and g_1 is the coefficient of $|x|^z$ in $g(x)$ as given in (3.8).

(7) It is simple to see how Feigenbaum's result can be recovered from (3.35)–(3.37). In the absence of logarithmic factors, $v_n = 0$. Then ϵ_n and w_n can both be taken to be zero and we are left with $u_{n+1} = \delta u_n$ only. This is the renormalization-group equation in Feigenbaum's case. It clearly implies that δ is independent of n and hence a universal constant. It is the presence of the v_n terms in

(3.36) that forces us to have a nontrivial additional rescaling ϵ_n . With this, even if $w_n=0$ for some value of n , $w_{n+1}=v_n+\dots$ does not vanish unless $v_n=0$. Further iteration renders Eq. (3.36) nontrivial again. This rescaling after each operation $I[\alpha, \tilde{g}_n]$ is analogous to the anomalous dimension of the wave-function renormalization that is present in quantum-field theory, while the usual rescaling by factor α corresponds to the canonical dimension.

This completes our renormalization-group analysis for the $p=1$ case. To summarize, Eqs. (3.43) and (3.47) give κ and τ in terms of some coefficients. These coefficients, namely β_1, g_1, a_4, d_2 as well as α and δ , are to be determined numerically. Alternatively, we can use (3.44) instead of (3.47) to determine κ .

IV. GENERALIZATION TO THE p -POSITIVE INTEGER CASE

A. The Jordan canonical form

Here we shall consider the p -positive integer case and the functional relation among α , δ , κ , and τ . First let us examine the functional space for $h(x)$, $t(x)$, and $s(x)$. From now on, $s(x)$ in Sec. III shall be referred to as $s_{p=1}(x)$ or simply $s_1(x)$.

A natural choice of the functional space to which $h(x), t(x)$ belong is the set of linear combinations of powers of x , that is, $|x|^{2n}x^{2m}$, $n=0,1,2,\dots$ and $m=0,1,2,\dots$. This means $h(x)=1+h_1|x|^2+h_2|x|^{2+2}+h_3|x|^{2+4}+\dots$ and similarly for $t(x)$. [Here the normalization of $h(x)$, $t(x)$ is taken to be $h(0)=t(0)=1$.] Let us refer to this as the functional space $F_z(p=0)$. To obtain the additional eigenvalue 1 for $s_1(x)$, we must enlarge the functional space to include the logarithmic factors. An examination of (3.26) clearly indicates that terms like $\ln(x^2)$ ($\beta_1|x|^2$ +terms with higher powers of $|x|$ must be included in addition to terms in $F_z(p=0)$). In order to define the operator $Q[\alpha]$ in (3.34), we see that terms like $[\ln(x^2)]^n$ ($|x|^{2z}$ +terms with higher powers of $|x|$), $n=0,1,2$, must also be included. If we want to keep higher-order terms in the renormalization-group equation (3.29), it is easy to see that a natural choice of the functional space to which $h(x)$, $t(x)$, and $s_1(x)$ belong is the set of the linear combinations of

$$[|x|^{2n}|\ln(x^2)|^m]x^{2k} \text{ with } n=0,1,2,\dots, \\ m=0,1,2,\dots,n, \text{ and } k=0,1,2,\dots$$

This is referred to as $F_z(p=1)$,

$$F_z(p=0) \subset F_z(p=1).$$

Now let us consider the case of positive integer p . A similar argument shows that for positive integer p , $F_z(p)$ is the set of linear combinations of $|x|^{2n}|\ln x^2|^m x^{2k}$ where $n=0,1,2,\dots$; $m=0,1,2,\dots,pn$; $k=0,1,2,\dots$. In terms of the operator L in (3.12), there is one eigenvalue of unity if the functional space is restricted to $F_z(p=0)$, as in the original work of Feigenbaum. In Sec. III we found two degenerate eigenvalues of unity because the

functional space has been enlarged to $F_z(p=1)$. In general, there will be $p+1$ degenerate eigenvalues of unity (hence $p+1$ marginal operators) for the functional space F_z (p =positive integer). It is straightforward to show that they appear in the Jordan canonical form

$$\begin{aligned} L[\alpha, t] &= t, \\ L[\alpha, s_1] &= s_1 + t, \\ L[\alpha, s_2] &= s_2 + s_1, \\ &\vdots \\ L[\alpha, s_{i+1}] &= s_{i+1} + s_i, \\ &\vdots \\ L[\alpha, s_p] &= s_p + s_{p-1}. \end{aligned} \quad (4.1)$$

To illustrate this, let us consider the $s_2(x)$ case

$$s_2(x) = \beta_2 |x|^2 [\ln(x^2)]^2 + \eta_1 |x|^2 \ln(x^2) + \dots,$$

where the dominant terms (as $x \rightarrow 0$) are kept. Using (3.9) we find that

$$\begin{aligned} L[\alpha, s_2] &= \beta_2 |x|^2 [\ln(x^2)]^2 \\ &\quad + (\eta_1 - 4\beta_2 \ln \alpha) |x|^2 \ln(x^2) + \dots \end{aligned}$$

so that consistency requires $L[\alpha, s_2] = s_2 + s_1$ and

$$-4\beta_2 \ln \alpha = \beta_1, \quad (4.2)$$

where β_1 is the coefficient in $s_1(x)$ as given in (3.48). Knowing β_1 , the normalization and the most important coefficient in $s_2(x)$ are determined. (On the other hand, η_1 is arbitrary. It depends on the choice of basis. If we want, we can choose it to be zero.) In fact, for, $m=1,2,\dots,p$,

$$s_m(x) = \beta_m |x|^2 [\ln(x^2)]^m + \dots \quad (4.3)$$

we have

$$-m\beta_m \ln(\alpha^2) = \beta_{m-1}$$

or

$$\beta_m = \frac{(-2 \ln \alpha)^{1-m}}{m!} \beta_1. \quad (4.4)$$

B. The renormalization-group equation

If the function to be considered has its extremum with the behavior $|x|^z |\ln(x^2)|^m$, then the corresponding $g_n(x)$ can be written as

$$g_n(x) = g(x) + u_n h(x) + w_n t(x) + \sum_{i=1}^m v_n^{(i)} s_i(x).$$

Substituting this into (3.5) and following the steps similar to those in Sec. III, we obtain the following set of renormalization-group equations, for large n :

$$\begin{aligned}
u_{n+1} &= \delta u_n + a_4 u_n v_n^{(1)} + \delta d_2 \epsilon_n u_n + \cdots, \\
w_{n+1} &= w_n + v_n^{(1)} + \epsilon_n + \cdots, \\
v_{n+1}^{(1)} &= v_n^{(1)} + v_n^{(2)} + \frac{c_6}{2} v_n^{(1)^2} + f_4 \epsilon_n v_n^{(1)} + \cdots, \\
v_{n+1}^{(2)} &= v_n^{(2)} + v_n^{(3)} + B^{(2)} v_n^{(1)} v_n^{(2)} \\
&\quad + C^{(2)} \epsilon_n v_n^{(2)} + \cdots,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\vdots \\
v_{n+1}^{(m-1)} &= v_n^{(m-1)} + v_n^{(m)} + B^{(m-1)} v_n^{(1)} v_n^{(m-1)} \\
&\quad + C^{(m-1)} \epsilon_n v_n^{(m-1)} + \cdots, \\
v_{n+1}^{(m)} &= v_n^{(m)} + B^{(m)} v_n^{(1)} v_n^{(m)} + C^{(m)} \epsilon_n v_n^{(m)} + \cdots,
\end{aligned}$$

where $B^{(i)}$ and $C^{(i)}$ ($i=2,3,\dots$) are defined by the following equations:

$$Q[\alpha, s_i, s_1] = B^{(i)} s_i(x) + \cdots \tag{4.6}$$

and

$$s_i(x) - x s_i'(x) = C^{(i)} s_i(x) + \cdots \tag{4.7}$$

Here the terms not displayed shall turn out to be small compared to the terms shown. Again, we can choose $\epsilon_n = -v_n^{(1)}$, so that $w_n = 0$ for all n . Let us define

$$A^{(i)} = B^{(i)} - C^{(i)}. \tag{4.8}$$

Before solving the renormalization-group equations to the order that is of interest to us, first let us show that

$$A^{(m)} = A^{(m-1)} = A^{(i)} = A^{(1)} = \frac{c_6}{2} + z - 1. \tag{4.9}$$

The rhs follows from the results of Sec. III, i.e., $B^{(1)} = c_6/2$ and $C^{(1)} = f_4$. The $C^{(i)}$ term is due to the re-scaling. It is easy to see that $C^{(i)} = 1 - z$ by investigating the most singular term of Eq. (4.7), i.e., $C^{(i)} = f_4 = 1 - z$. In the limit $x \rightarrow 0$, we find that Eq. (4.6) becomes

$$\begin{aligned}
Q[\alpha, s_i, s_1] &= - \left[\frac{s_1'(1) + g''(1) s_1(0)}{\alpha^{z-1}} \right] s_i(x) + \cdots \\
&= \frac{c_6}{2} s_i(x) + \cdots,
\end{aligned} \tag{4.10}$$

where the relation

$$c_6 = - \frac{2[s_1'(1) + g''(1) s_1(0)]}{\alpha^{z-1}}$$

is given in Appendix C. This completes the proof for (4.9).

Using Eqs. (3.43) and (3.44), Eq. (4.5) becomes, for large n and $p = m$,

$$\begin{aligned}
u_{n+1} &= u_n \delta \left[1 + \frac{\tau}{\kappa} v_n^{(1)} \right] + \cdots, \\
v_{n+1}^{(1)} &= v_n^{(1)} + v_n^{(2)} + \frac{v_n^{(1)^2}}{\kappa} + \cdots, \\
&\vdots \\
v_{n+1}^{(i)} &= v_n^{(i)} + v_n^{(i+1)} + \frac{v_n^{(1)} v_n^{(i)}}{\kappa} + \cdots, \\
&\vdots \\
v_{n+1}^{(m)} &= v_n^{(m)} + \frac{v_n^{(1)} v_n^{(m)}}{\kappa} + \cdots,
\end{aligned} \tag{4.11}$$

where $i=1,2,3,\dots,m$. The solution to this set of equations is straightforward; for $v_n^{(m)} \neq 0$, we have

$$\begin{aligned}
v_n^{(1)} &\approx - \frac{m\kappa}{n}, \\
v_n^{(2)} &\approx - \frac{m(m-1)\kappa}{n^2}, \\
&\vdots \\
v_n^{(i)} &\approx - \frac{m! \kappa}{(m-i)! n^i},
\end{aligned} \tag{4.12}$$

$$u_{n+1} \approx u_n \delta \left[1 - \frac{m\tau}{n} \right],$$

which give

$$\begin{aligned}
\alpha_n &\approx \alpha \left[1 - \frac{m\kappa}{n} \right], \\
\delta_n &\approx \delta \left[1 - \frac{m\tau}{n} \right].
\end{aligned} \tag{4.13}$$

This verifies the numerical results (2.15) for $p = m$. Two comments are in order.

(1) One can easily verify that the terms neglected in Eqs. (4.5) and (4.11) are down by powers of n and hence can be ignored.

(2) The result [(4.11)–(4.13)] is independent of the choice of s_i [i.e., $s_i(x)$ can be replaced by $s_i(x) + r_i s_{i-1}(x)$ for $i=2,3,\dots$].

C. Functional relations

In the derivation of the Jordan canonical form (3.25) we obtain

$$s_1(x) = -\alpha_z \left[\frac{\partial \alpha_z}{\partial z} \right]^{-1} \frac{\partial g_z(x)}{\partial z}, \tag{4.14}$$

which implies that κ and τ are connected to changes of $g_z(x)$, $\alpha(z)$, $\delta(z)$ in the z direction. Here we shall elucidate this relationship.

Consider a function with extremum of the power $z - \epsilon$,

$$f(x) = |x|^{z-\epsilon} \quad (4.15)$$

$$= |x|^z \exp(-\epsilon \frac{1}{2} \ln(x^2))$$

$$= |x|^z \sum_{k=0}^{\infty} \frac{(-\epsilon \frac{1}{2} \ln(x^2))^k}{k!} \quad (4.16)$$

From (4.15) we see, for this type of function with large n and small ϵ , that

$$\alpha_n \approx \alpha(z - \epsilon)$$

$$\approx \alpha(z) \left[1 - \frac{\epsilon}{\alpha(z)} \frac{\partial \alpha(z)}{\partial z} \right], \quad (4.17)$$

$$\delta_n \approx \delta(z - \epsilon)$$

$$\approx \delta(z) \left[1 - \frac{\epsilon}{\delta(z)} \frac{\partial \delta(z)}{\partial z} \right], \quad (4.18)$$

where only leading terms in ϵ are kept. These also indicate that for large n , α_n and δ_n are independent of n , as must be the case for the function (4.15). Now, let us consider (4.16), which is equivalent to (4.15). Using the Sterling formula for large k , we have

$$f(x) \approx |x|^z \sum_k \left[\frac{-\epsilon \frac{1}{2} \ln(x^2)}{k} \right]^k k^{-1/2}, \quad (4.19)$$

which is a good approximation only for the terms with large k . For a given large n the relevant value of the x variable is x/α^n where $|x| \approx [0, 1]$. Therefore, the magnitude of an individual term in the sum in (4.19) is, for large k ,

$$\sigma(k) \approx \left[\frac{+\frac{1}{2}\epsilon \ln(\alpha^{2n})}{k} \right]^k k^{-1/2}. \quad (4.20)$$

The term that has the largest contribution to (4.19), for a given n , is given by

$$\frac{\partial \sigma(k)}{\partial k} = 0$$

or

$$k \approx n \epsilon \ln \alpha. \quad (4.21)$$

For sufficiently small ϵ , this term in fact dominates. For this value of k , we know, from (4.13),

$$\alpha_n \approx \alpha(z) \left[1 - \frac{k\kappa}{n} \right]$$

$$\approx \alpha(z)(1 - \epsilon\kappa \ln \alpha), \quad (4.22)$$

$$\delta_n \approx \delta(z) \left[1 - \frac{k\tau}{n} \right]$$

$$\approx \delta(z)(1 - \epsilon\tau \ln \alpha). \quad (4.23)$$

Comparing (4.17) with (4.22) and (4.18) with (4.23), we obtain

$$\frac{\partial \alpha(z)}{\partial z} = \kappa(z) \alpha(z) \ln \alpha(z), \quad (4.24)$$

$$\frac{\partial \delta(z)}{\partial z} = \tau(z) \delta(z) \ln \alpha(z), \quad (4.25)$$

where the z dependence in $\alpha(z)$, $\delta(z)$, $\kappa(z)$, and $\tau(z)$ are explicitly displayed. Of course, we can combine them to write

$$\frac{\partial \alpha}{\partial \delta} = \frac{\kappa \alpha}{\tau \delta}. \quad (4.26)$$

D. Numerical results

Now we are ready to calculate κ and τ using the renormalization-group analysis. To obtain the eigenvalues of the linear operator L , we consider the following truncation of our functional space (for $z=2$, $p=1$ case):

$$V(x) = l_1 + l_2 x^2 + l_3 x^2 \ln(x^2) + l_4 x^4 + l_5 x^4 \ln(x^2) + l_6 x^4 [\ln(x^2)]^2 + l_7 x^6 + \dots \quad (4.27)$$

Since we truncate the function space $F_{z=2}(p=1)$, we shall obtain two eigenvectors whose eigenvalues are very close to 1, instead of two degenerate eigenvalues that form a Jordan canonical form of degree 2. The difference of the two eigenvectors is essentially $s(x)$. We can obtain $s(x)$ either by this method or, equivalently, by using Eq. (3.25). In Table V we give the result for the operators $h(x)$, $t(x)$, $s(x)$, and their eigenvalues. Here 14 terms in (4.27) are used. It is easy to check numerically that some of the coefficients in Table V are very stable; in particular the coefficients of x^2 , $x^2 \ln(x^2)$ are very stable. The coefficient $l_1=1$ determines the normalization of $h(x)$ and $t(x)$. The choice of $l_1=0$ for $s(x)$ fixes the choice of $s(x)$.

For $z=2$ the coefficient g_1 in $g(x)$ is given by $g_1 = -1.5276$. Taking $\beta_1 = 1.8728$ from Table V, we find, from Eq. (3.47), $\kappa = -0.4445$. In Table VI the various coefficients in Eqs. (3.33) and (3.34) are numerically determined using the eigenvectors obtained from Table V. To see the accuracies of these coefficients, we check the theoretical relations (3.38), (3.45), and (3.46) with the numerical values in Table VI. This shows that the numbers in Table VI are typically reliable to 3 or 4 significant figures. Using (3.43) and (3.44) we obtain

$$\kappa = -0.4445$$

and

$$\tau = 0.3695.$$

Of course, better accuracy can be obtained if we include more terms in Eq. (4.27). κ and τ are stable when we vary the functional space in a way such that terms in F_p ($p \geq 2$) are excluded. However, some of the coefficients of the higher terms may become unstable. This implies that an optimum choice of the functional space may not have been achieved.

We can also obtain κ and τ by using Eqs. (4.24) and (4.25). Comparing $z=2.0001$ and 2.0000 gives the same κ and τ . For $z=1.5$, $\alpha=3.3887$, $\delta=3.8006$, $\kappa=-0.7622$, and $\tau=0.4206$.

TABLE V. Eigenvalues and eigenvectors for the relevant operator $h(x)$ and the marginal operators $t(x)$ and $s_1(x)$ obtained from numerical solution to the Eqs. (3.13), (3.14), and (3.25).

Operator Eigenvalue	$h(x)$ 4.669 20	$t(x)$ 1.002 18	$s_1(x)$ 0.997 80
1	1.000 00	1.0000	0.0
x^2	-0.325 65	1.5274	-0.681 63
$x^2 \ln(x^2)$	1.3×10^{-6}	2.406×10^{-5}	1.8728
x^4	-0.033 78	-0.1965	0.168 18
$x^4 \ln(x^2)$	-4.656×10^{-2}	-0.034 52	-0.254 55
$x^4 [\ln(x^2)]^2$	3.85×10^{-4}	3.1510×10^{-3}	-8.76×10^{-4}
x^6	0.231 56	0.392 92	2.6078
$\vdots$	$\vdots$	$\vdots$	$\vdots$

V. FURTHER GENERALIZATION

A. The $q \neq 0$ case

To generalize the result of Secs. III and IV to the p =arbitrary real number case is very difficult because the functional space for the operator $s_p(x)$ in general is not known. With some assumptions on the functional space and properties of $s_p(x)$, one can make an argument for the general p case. However, in our viewpoint this is tantamount to assuming the answer. The numerical data for the small p case is very good. So here we shall assume that (2.15) is true for small real number p . We then study some of the consequences of this assumption; numerical computation can then be used to check the consequences.

The idea is to repeat the method developed in Sec. IV C for logarithmic terms,

$$\Delta f \approx |x|^z |\ln(x^2)|^{p+\epsilon} = |x|^z |\ln(x^2)|^p \sum_{k=0}^{\infty} \frac{[\epsilon \ln |\ln(x^2)|]^k}{k!}. \quad (5.1)$$

For large n , the left-hand side gives the behavior

$$\alpha_n \approx \alpha \left[1 - \frac{\kappa(p+\epsilon)}{n} \right], \quad (5.2)$$

$$\delta_n \approx \delta \left[1 - \frac{\tau(p+\epsilon)}{n} \right]. \quad (5.3)$$

Clearly, consistency requires the rhs to have the same behavior. Here we can use the same method as in Sec. IV C. For simplicity, let us consider the case where $p=0$. Then intuitively we expect, for $\Delta f \approx |x|^z [\ln |\ln(x^2)|]^l$,

$$\lambda_{\infty} - \lambda_n \approx \delta^{-n} (\ln n)^{l\xi} \quad (5.4)$$

for large n , or

$$\delta_n \approx \delta \left[1 - \frac{l\xi}{n \ln n} \right], \quad (5.5)$$

where ξ is some constant, and similarly,

$$\alpha_n \approx \alpha \left[1 - \frac{l\gamma}{n \ln n} \right]. \quad (5.6)$$

To obtain the large n behavior for (5.1), we must determine the dominant k term for a fixed, large n . It is straightforward to show that the dominant term has k given by $k \approx \epsilon \ln[-\ln(x^2)]$. At the n th period doubling the relevant value of x is $x\alpha^{-n}$ so that

$$k \approx \epsilon \ln[2n \ln \alpha - \ln(x^2)] \approx \epsilon [\ln n + \ln \ln(\alpha^2)]. \quad (5.7)$$

This means

TABLE VI. Numerical values of the coefficients in the renormalization-group equation (3.35)–(3.37). They are expected to have an uncertainty of the order of 10^{-4} . The coefficients are defined in Eqs. (3.33) and (3.34).

$a_1=4.771\,65$	$b_1=2.123\,59$	$c_1=-1.45 \times 10^{-5}$
$a_2=-6.76 \times 10^{-5}$	$b_2=1.896\,98$	$c_2=-3.567 \times 10^{-5}$
$a_3=-2.971\,54$	$b_3=3.51 \times 10^{-4}$	$c_3=-9.444 \times 10^{-5}$
$a_4=6.106\,94$	$b_4=1.624\,86$	$c_4=0.782\,25$
$a_5=6.521\,28$	$b_5=1.278\,08$	$c_5=8.398 \times 10^{-5}$
$a_6=-4.117\,55$	$b_6=3.971\,57$	$c_6=2.499\,58$
$d_1=0.404\,911$	$e_1=-0.360\,59$	$f_1=-4.89 \times 10^{-5}$
$d_2=0.476\,548$	$e_2=0.516\,992$	$f_2=-7.50 \times 10^{-6}$
$d_3=0.809\,838$	$e_3=0.278\,433$	$f_3=-1.20 \times 10^{-6}$
$d_4=-1.556\,57$	$e_4=1.291\,14$	$f_4=-1.000\,03$

$$\begin{aligned}\delta_n &\approx \delta \left[-\frac{\epsilon \zeta [\ln n + \ln \ln(\alpha^2)]}{n \ln n} \right] \\ &\approx \delta \left[1 - \frac{\epsilon \zeta}{n} \right]\end{aligned}\quad (5.8)$$

and similarly

$$\alpha_n \approx \alpha \left[1 - \frac{\epsilon \gamma}{n} \right]. \quad (5.9)$$

Comparing with (5.2) and (5.3) for $p=0$, we obtain

$$\zeta = \tau$$

and

$$\gamma = \kappa.$$

This supports the formulas (1.2) and (1.3). We can check this by numerical iterations of some functions which have extrema of the form $\Delta f \approx |x|^z \{ \ln[-a \ln(x^2)] \}^q$. From (5.7) and (5.8) we see that the approach is extremely slow,

$$\tau \approx \zeta \left[1 + \frac{\ln[a \ln(\alpha^2)]}{\ln n} \right]. \quad (5.10)$$

If we consider the case in (5.7), i.e., $a=1$ and $z=2$, the correction term is quite substantial. Since the computational time increases like 2^n , the maximum n that is feasible via numerical iteration is $n \approx 20$. Then ζ will still be 20% away from τ . In Figs. 6 and 7, the value of κ and τ extracted from functions with $p=0$ and various values of q are shown. The functions used have the form

FIG. 6. Value of τ extracted from functions of the type (5.11) with different q values where $p=0$. The curves are obtained from the following fits: (1) For the solid curves with $q=0.1, 1.0, 2.0$, the formula (5.12) is used; (2) For the dashed curve Eq. (5.13) is used. See the description in the text.

FIG. 7. Value of κ extracted from functions of the type (5.11) with different q values where $p=0$. The curves are obtained from fits that are described in the text.

$$f(x) \approx \lambda \{ 1 - (2x-1)^2 [c_1 |\ln |\ln(2x-1)|^2| + c_2] \}^q, \quad (5.11)$$

where (q, c_1, c_2) takes the value $(2, \frac{1}{2}, 1)$, $(1, 1, \frac{1}{2})$, and $(0.1, \frac{1}{2}, \frac{1}{2})$. The τ value is extracted from the following fitting formula:

$$\frac{\delta_n}{\delta} = 1 - \frac{\tau}{n} \frac{1}{\ln[n \ln(\alpha^2)]} \left[1 + \frac{A}{n} + \frac{B}{n^2} \right], \quad (5.12)$$

where the α_n data are generated numerically for $n=1, 2, \dots, 17$ (the solid curves in Fig. 6). We also consider the case where $a=+0.544\dots$ such that $\ln[a \ln(\alpha^2)] \approx 0$ so that the $(\ln n)^{-1}$ correction is suppressed. The dashed curve in Fig. 6 is for the function

$$f(x) = \lambda \{ 1 - (2x-1)^2 \ln[-a \ln(2x-1)^2] \}^{0.1}, \quad (5.13)$$

where $q=0.1$ and $a=0.544\dots$. The τ value is extracted from the fitting formula:

$$\frac{\delta_n}{\delta} = 1 - \frac{\tau}{n \ln n} \left[1 + \frac{A'}{n} + \frac{B'}{n^2} \right].$$

Formulas similar to (5.12) and (5.13) are used to extract the κ value in Fig. 7. We see that Figs. 6 and 7 do provide clean support for the asymptotic form (1.2) and (1.3). Of course, the accuracy is not as good as that for the $p \neq 0$, $q=0$ case, as expected from the slow approach to asymptotic behavior as n becomes large. For functions whose extremum at $x=0$ has the form

$$f(x) \approx |x|^z \left| \ln |\ln | \dots | \ln x^2 | \dots | \right|^r$$

the asymptotic behavior becomes exceedingly hard to

reach. However, by repeating the above analytic argument, it clearly indicates that no new universal constants (besides α , δ , τ , and κ) will emerge.

B. The general symmetric case

The above analysis allows us to extend our discussion to the general symmetric case. In fact, both the derivation and the result are very simple. Let us consider functions whose (symmetric) extremum at $x=0$ has the form of Eq. (1.8),

$$f(x) \approx |x|^z b(x). \quad (1.8)$$

If $b(x)$ modifies the form $|x|^z$ by more than a power, the universal period-doubling approach to chaos will be lost. If $b(x)$ modifies the form by a power, we can simply redefine the value of z . Hence, we need to consider only the case where, for any infinitesimal positive constant ν , as $x \rightarrow 0$,

$$|x|^\nu b(x) \rightarrow 0, \quad (1.9)$$

$$|x|^{-\nu} b(x) \rightarrow \infty.$$

At the n th bifurcation the relevant value of x^2 is given by $x_n^2 \approx \alpha^{-2n}$. In general, the effective power of $|x|$ in the function $f(x)$ in Eq. (1.8) is given by

$$\begin{aligned} z(x) &= \frac{x}{f(x)} \frac{\partial f(x)}{\partial x} \\ &= z + \frac{\partial \ln b(x)}{\partial \ln |x|}. \end{aligned}$$

At the n th bifurcation, for asymptotic n ,

$$z(n) = z + \epsilon(n),$$

where

$$\epsilon(n) = \frac{\partial \ln b(x)}{\partial \ln |x|} \bigg|_{|x_n| \approx \alpha^{-n}}. \quad (1.12)$$

Now, the effective α and δ are the corresponding values at the effective power $z(n)$. Following Eqs. (4.17) and (4.18), we have, for asymptotically large n ,

$$\begin{aligned} \alpha_n &= \alpha(z(n)) \approx \alpha(z) \left[1 + \frac{\epsilon(n)}{\alpha(z)} \frac{\delta \alpha(z)}{\delta z} \right], \\ \delta_n &= \delta(z(n)) \approx \delta(z) \left[1 + \frac{\epsilon(n)}{\delta(z)} \frac{\partial \delta(z)}{\partial z} \right], \end{aligned} \quad (5.14)$$

where terms of $O(\epsilon^2)$ are neglected. Clearly, we expect, for any fixed ν , as $x \rightarrow 0$ and $n \rightarrow \infty$,

$$\frac{\epsilon(n)}{\nu} \rightarrow 0.$$

Using (1.4) and (1.5), we obtain the final result

$$\alpha_n \approx \alpha [1 + \kappa \epsilon(n) \ln \alpha], \quad (1.10)$$

$$\delta_n \approx \delta [1 + \tau \epsilon(n) \ln \alpha]. \quad (1.11)$$

For example, if $b(x)$ has the following form:

$$b(x) \approx |\ln(x^2)|^p |\ln |\ln(x^2)||^q |\ln |\ln |\ln(x^2)|||^r \dots$$

then

$$\epsilon(n) = \frac{-1}{\ln \alpha} \left[\frac{p}{n} + \frac{q}{n \ln(n)} + \frac{r}{n \ln(n) \ln \ln(n)} + \dots \right].$$

In Eq. (5.14) we have consistently neglected terms of the order ϵ^2 (or n^{-2}). To summarize, for functions whose extremum at $x=0$ has the form

$$\Delta f_\lambda \approx |x|^z |\ln x^2|^p |\ln |\ln x^2||^q |\ln |\ln |\ln x^2|||^r \dots, \quad (5.15)$$

where $z, p, q, r, \dots$ are all real numbers and $z > 1$, their behavior of period-doubling bifurcations is quantitatively universal as they approach the chaotic point, i.e., as λ approaches the critical λ_c . At the n th bifurcation (n is asymptotically large) the value of λ_n behaves as

$$\lambda_c - \lambda_n \approx \delta^{-n} n^p (\ln n)^q (\ln \ln n)^r \dots \quad (5.16)$$

and the typical distance away from the extremum at $\lambda = \lambda_n$ is

$$d_n \approx \alpha^{-n} n^{\kappa p} (\ln n)^{\kappa q} (\ln \ln n)^{\kappa r} \dots \quad (5.17)$$

It is amazing that no new universal constants are introduced when $q \neq 0$, $r \neq 0, \dots$. The above derivation is very simple and general. The reader may ask, with such a derivation, why we need the renormalization-group analysis discussed earlier, since the RG analysis is more complicated and less general. The reason is that the simple derivation given in this subsection is only a heuristic one. Specifically, we observe that there is more than one way to define the effective power $z(x)$, and hence $\epsilon(n)$. The $\epsilon(n)$ defined in Eq. (1.12) is justified only after the detailed RG analysis has been performed [e.g., the effective power $z(x)$ defined by $f(x) \approx |x|^{z(x)}$ is incorrect]. The renormalization-group analysis plus the above derivation give us the universal quantitative properties of the general symmetric case.

C. The asymmetric case

For the sake of completeness, let us consider other type of functions that can exhibit cascades of period doubling, namely, functions that have asymmetric extrema, as those given in (1.13). Here we shall briefly summarize the universal metric properties of such functions.⁴ It is clear that there are two universal functions $g_1(x)$ and $g_2(x)$ such that

$$\alpha_1 g_1(g_1(x/\alpha_1)) = g_2(x),$$

$$\alpha_2 g_2(g_2(x/\alpha_2)) = g_1(x).$$

We can introduce $\sigma = \alpha_1 \alpha_2$ and combine them into one equation for the universal function $\bar{g}(x)$,

$$\sigma \bar{g}(\bar{g}(\bar{g}(x/\sigma))) = \bar{g}(x), \quad (5.18)$$

where $\bar{g}(x)$ and σ are both functions of z and ϵ ,

$=\tilde{s}(x)+rt(x)$ for some constant r . Then

$$Q[\alpha, h, \tilde{s}] = \tilde{a}_4 h(x) + \tilde{b}_4 t(x) + \tilde{c}_4 \tilde{s}(x) + \cdots \quad (\text{A2})$$

and we want to find the relationship of the old set of coefficients to the new set of coefficients. Recall

$$Q[\alpha, h, t] = a_2 h(x) + b_2 t(x) + c_2 s(x) + \cdots, \quad (\text{A3})$$

where $c_2=0$ since $Q[\alpha, h, t]$ has no logarithmic terms. Since the operator $Q[\alpha, h, s]$ is linear in $s(x)$,

$$\begin{aligned} Q[\alpha, h, s] &= Q[\alpha, h, \tilde{s}] + rQ[\alpha, h, t] \\ &= (a_4 + ra_2)h(x) + (\tilde{b}_4 + rb_2 - r\tilde{c}_4)t(x) \\ &\quad + \tilde{c}_4 \tilde{s}(x) + \cdots \end{aligned} \quad (\text{A4})$$

Comparing (A1) and (A4) we find

$$a_4 = \tilde{a}_4 + ra_2, \quad (\text{A5})$$

$$b_4 = \tilde{b}_4 + rb_2 - r\tilde{c}_4. \quad (\text{A6})$$

It is straightforward to generalize this argument to obtain the following results. $a_1, b_1, a_2, b_2, a_3, b_3, c_4$, and c_5 as well as $d_1, d_2, d_3, e_1, e_2, e_3, f_1, f_2, f_3$, and f_4 are invariant under the change of $s(x)$, while

$$\begin{aligned} \tilde{a}_4 &= a_4 - ra_2, \\ \tilde{a}_5 &= a_5 - ra_3, \\ \tilde{a}_6 &= a_6 - 2ra_5 + r^2 a_3, \\ \tilde{b}_4 &= b_4 - rb_2 + rc_4, \\ \tilde{b}_5 &= b_5 - rb_3 + rc_5, \\ \tilde{b}_6 &= b_6 - 2rb_5 - 2r^2 c_5 + r^2 b_3 + rc_6, \\ \tilde{c}_6 &= c_6 - 2rc_5, \\ \tilde{d}_4 &= d_4 - rd_3, \\ \tilde{e}_4 &= e_4 - re_3 + rf_4. \end{aligned} \quad (\text{A7})$$

APPENDIX B

The operator $I[\alpha_n, g_n]$ in Eq. (3.27) can be written as a product of two operators,

$$I[\alpha_n, g_n] = R[\epsilon_n, I[\alpha, g_n]], \quad (\text{B1})$$

where

$$R[\epsilon, f(x)] = (1+\epsilon)f\left[\frac{x}{1+\epsilon}\right] \quad (\text{B2})$$

and

$$I[\alpha, f(x)] = -\alpha f(f(-x/\alpha)). \quad (\text{B3})$$

Note that

$$R[\epsilon, f_1 + f_2] = R[\epsilon, f_1] + R[\epsilon, f_2]. \quad (\text{B4})$$

Also R and I commute, i.e.,

$$\begin{aligned} I[\alpha, R[\epsilon, f]] &= I\left[\alpha, (1+\epsilon)f\left[\frac{x}{1+\epsilon}\right]\right] \\ &= -\alpha(1+\epsilon)f\left[f\left[\frac{-x}{(1+\epsilon)\alpha}\right]\right] \\ &= R\left[\epsilon, -\alpha f\left[f\left[\frac{-x}{\alpha}\right]\right]\right] \\ &= R[\epsilon, I[\alpha, f]]. \end{aligned} \quad (\text{B5})$$

In Sec. III Eqs. (3.35)–(3.37) are obtained by evaluating $I[\alpha, R[\epsilon_n, g_n]]$. We can obtain the same equations by evaluating $R[\epsilon_n, I[\alpha, g_n]]$. Comparing the two sets of equations and equating the coefficients of the terms to quadratic powers, i.e., to $u_n^2, u_n v_n, u_n w_n, v_n w_n, v_n^2, w_n^2 \epsilon_n u_n, \epsilon_n v_n, \epsilon_n w_n$, we obtain the relations (3.45). The calculation is straightforward. In particular, it is easy to see that the coefficients $a_2 = b_3 = c_5 = 0$ since the corresponding terms are absent in $R[\epsilon_n, I[\alpha, g_n]]$.

APPENDIX C

Let us differentiate (3.25) with respect to x , divide the resulting equation by $|x|^{z-1}$, and then take the $x \rightarrow 0$ limit. This straightforward procedure gives the following relation:

$$s'(1) + g''(1)s(0) = \left[(z-1) - \frac{2\beta_1 \ln \alpha}{g_1}\right] \alpha^{z-1}, \quad (\text{C1})$$

where g_1 is the coefficient of $|x|^z$ in $g(x)$ as given in (3.8) and β_1 is the coefficient of $|x|^z \ln(x^2)$ in $s(x)$,

$$s(x) = \beta_1 |x|^z \ln(x^2) + \cdots \quad (\text{C2})$$

Next, let us turn our attention to $Q[\alpha, s, s]$,

$$Q[\alpha, s, s] = c_6 s(x) + \cdots$$

Equating the coefficient of the $|x|^z \ln(x^2)$ term, we obtain

$$\frac{-2[s'(1) + g''(1)s(0)]}{\alpha^{z-1}} = c_6. \quad (\text{C3})$$

We observe that the above procedure implicitly assumes that all irrelevant operators do not contain the term $|x|^z \ln(x^2)$. This is because any eigenoperator that contains the $|x|^z \ln(x^2)$ term should have eigenvalue one (see Sec. IV B). Therefore

$$\kappa^{-1} = -\left[\frac{c_6}{2} + z - 1\right] = \frac{-2\beta_1 \ln \alpha}{g_1}. \quad (\text{C4})$$

Recall that we can choose r such that

$$\begin{aligned} s(x) &= -\alpha_z \left[\frac{\delta \alpha_z}{\delta z}\right]^{-1} \frac{\delta g_z}{\delta z}(x) \\ &= -\alpha_z \left[\frac{\delta \alpha_z}{\delta z}\right]^{-1} \left[\frac{g_1}{2} |x|^z \ln(x^2) + \cdots\right], \end{aligned} \quad (\text{C5})$$

$$\bar{g}(x) = \begin{cases} 1 + (1+\epsilon)g_1 |x|^z + \cdots, & x > 0 \\ 1 + (1-\epsilon)g_1 |x|^z + \cdots, & x < 0. \end{cases} \quad (5.19)$$

Of course, for $\epsilon=0$, $\bar{g}(x)=g_z(x)$ and $\sigma(z,\epsilon=0)=\alpha^2(z)$. For $\epsilon \neq 0$, besides $\bar{g}(z,\epsilon,x)$ and $\sigma(z,\epsilon)$, there is a corresponding set of eigenvectors and eigenvalues, which can be obtained from

$$\begin{aligned} & \sigma \{ \bar{g}'(\bar{g}(\bar{g}(x/\sigma))) \\ & \times [\bar{g}'(\bar{g}(x/\sigma))][\bar{g}'(x/\sigma)]\bar{h}(x/\sigma) \\ & + \bar{h}(\bar{g}(x/\sigma))] \\ & + \bar{h}(\bar{g}(\bar{g}(x/\sigma))) \} = \lambda \bar{h}(x). \end{aligned} \quad (5.20)$$

For each value of ϵ , there are two solutions to Eq. (5.18): $g_1(x)$ and $g_2(x)$. For each $g_i(x)$, $i=1,2$, there is a set of eigenvectors $h_{i,j}$ with corresponding eigenvalues λ_j , $i=1,2$ and $j=1,2,3,\dots$. For $\epsilon=0$ the leading operators coincide with $h(x)$, $\bar{h}_{11}=\bar{h}_{21}=h(x)$, and the corresponding eigenvalue $\lambda(z,\epsilon=0)=\delta^2$. In fact, it turns out that the universal functions $g_i(x)$ and the eigenoperators are both functions of the rescaled x only; to explain this, let us consider $z=2$, then

$$g_1(x) = \begin{cases} 1 + \sum_i g_1^{(i)} [(1+\epsilon)x^2]^i, & x > 0 \\ 1 + \sum_i g_1^{(i)} [(1-\epsilon)x^2]^i, & x < 0 \end{cases}$$

where the coefficients $g_1^{(i)}$ are the same for $x \geq 0$. $g_2(x)$ and $h_{ij}(x)$ have similar behaviors. As an example, for $\epsilon=0.1$, we have

$$\sigma(z=2, \epsilon=0.1) = (2.504\,898\,4)^2, \quad (5.21)$$

$$\lambda(z=2, \epsilon=0.1) = (4.677\,318\,9)^2. \quad (5.22)$$

We leave it as an exercise for the reader to consider the case where the extremum has both the logarithms and the asymmetry.

VI. REMARKS AND CONCLUSIONS

One-dimensional maps that exhibit repeated period-doubling bifurcations are being analyzed in some detail here. It is shown that the universal metric properties of this period-doubling behavior are in general determined by four universal constants, α , δ , κ , and τ ; the general symmetric case is obtained by a simple derivation (Sec. VB). The justification of this intuitively clear derivation comes from the more careful and detailed analysis in Secs. III and IV; here the RG method plays a crucial role. The RG method is useful in two ways. It allows us to analytically derive two functional relations among α , δ , κ , and τ . This provides a valuable insight into the underlying structure of the universality of 1D maps. The RG method also makes the numerical determination of the universal constants much easier. To determine κ, τ numerically without the use of RG, we simply pick a function and start to iterate it. The computational time goes like 2^n as the accuracy of κ and τ improve like n^{-c} for some constant c (in contrast, the accuracy of α and δ improve like a^{-n} for

some constant a). If we use the RG method the computational time goes like n^2 , which is clearly much more efficient. For example, if we want κ and τ to be determined to four significant figures, the RG method requires only a few seconds of central-processing unit time on the IBM 3081 while the brute-force approach requires approximately an hour for each function to be iterated.

The results of the above analysis are also applicable to the study of band bifurcation on the chaotic side, i.e., $\lambda > \lambda_c$. This is a reflection of the simplicity of 1D maps in comparison to higher-dimensional maps. It will be interesting to consider the effect of noise on systems with logarithmic modifications. We observe that all functions considered in our analysis are at least once differentiable.

The RG analysis hinges on one important feature. The $g(x)$ function and hence the operator L belong to the "power-of- x " space. We find that the space in which solutions to L exist is in general larger than the space in which L belongs, i.e., solutions to the equation with the operator L can contain functional forms that are absent in the power-of- x space. Such a feature is quite common in physics. For example, consider Newton's equation for the earth moving around the sun. The differential operator is spherically symmetric and circles are clearly solutions of the equation, but so are elliptical orbits. In quantum-field theory there are ample examples where the solution of an equation spontaneously breaks the symmetry of the equation.

Our analysis clearly raised the question concerning functional space. In fact, this issue is already present in the Feigenbaum analysis. For a general z the functional space for $g_z(x)$ is not known. In general, a necessary condition on the functional space F is that, given a function $g_z(x)$ defined in F , the operation on the function $g_z(x)$ does not take it out of F . However, $g_z(x)$ of the form

$$g_z(x) = 1 + g_1 |x|^z + g_2 |x|^{2z} + g_3 |x|^{3z} + \cdots$$

(for $z \neq 2$) is in general inappropriate in the determination of $\alpha(z)$. This issue of functional space clearly requires further investigation. A better understanding of the space F_p and its generalization $F_{p,q,\dots}$ will be useful.

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APPENDIX A

If $s(x)$ is a solution of the equation

$$L[\alpha, s] = s(x) + t(x), \quad (3.25)$$

where $t(x)$ is given by Eq. (3.14), so is any linear combination of $s(x)$ and $t(x)$. Let us consider the equation

$$\begin{aligned} Q[\alpha, h, s] &= Q[\alpha, s, h] \\ &= a_4 h(x) + b_4 t(x) + c_4 s(x) + \cdots \end{aligned} \quad (A1)$$

Suppose we change to another basis $s(x)$, where $s(x)$

where Eq. (3.8) is used. Combining (C2), (C4), and (C5), we obtain

$$\beta_1 = -\alpha_z \left[\frac{\delta \alpha_z}{\delta z} \right]^{-1} \frac{g_1}{2} \quad (C6)$$

or

$$\frac{1}{\alpha_z} \frac{\delta \alpha_z}{\delta z} = \kappa \ln \alpha_z, \quad (C7)$$

where (C7) is obtained by considering only the $p=1$ case.

¹See, e.g., P. Collet and J.-P. Eckmann, *Iterated Maps of the Interval as Dynamical Systems* (Birkhauser, Cambridge, Mass., 1980); J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, *Phys. Rep.* **92**, 45 (1982).

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