Induced Hopf term in the nonlinear σ model

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We study the statistics of solitons in the $O(3)$ nonlinear σ model in 2+1 dimensions coupled to fermions with an arbitrary dispersion relation. We show that P and T violation is not a sufficient condition for the appearance of anyons in the spectrum. We give a simple formula for the coefficient of the induced Hopf term, and formulas for the induced Chern-Simons term and fermion number, where appropriate. They depend only on the high- and low-momentum behavior of the fermion dispersion relation, and are stable against perturbations (such as anisotropies) of the dispersion relation which preserve its asymptotic behavior. We apply our results to models of high- T_c materials, and find that, even with P and T violation, anyons in general do not emerge. In a relativistic model, the soliton is always a fermion. We discuss a hypothetical model in which anyons are generated.

I. INTRODUCTION

Recently, the theoretical possibility of fractional statistics¹⁻³ in 2+1 dimensions has attracted a lot of attention. Fractional statistics is characterized by a parameter $\theta \in [-1, 1]$ defined modulo 2. Bose statistics corresponds to $\theta = 0$, Fermi statistics to $\theta = 1$, and fractional statistics to nonintegral values of θ . Generically, the presence of anyons² (excitations obeying fractional statistics) implies a breakdown (explicit or spontaneous) of time (T) reversal and parity (P) at the microscopic level.⁴ However, P and T violations do not necessarily imply that anyons appear in the spectrum. In this paper, we will investigate under what general conditions anyons may appear in theories involving the $O(3)$ nonlinear σ ($NL\sigma$) model coupled to fermions.

The $NL\sigma$ model appears in the description of many two-dimensional systems. For instance, it has been demonstrated that the $NL\sigma$ model accounts very well for the behavior of a class of high- T_c materials in their insulating (low doping) phase.⁵ The $NL\sigma$ model describes spin waves (magnons) in the Néel state of Heisenberg antiferromagnets.⁶ The $NL\sigma$ model may also describe the order parameter for thin films in A -type phases of superfluid ^3He , and couples to the spins of helium nuclei.⁷ It was first pointed out by Wilczek and Zee that the solitons of the $NL\sigma$ model are anyons if a topological term (the Hopf term⁸) is present,^{9,10} with a fractional coefficient, in the action. It is well known that a gas of charged anyons is superconducting.¹¹ We will assume that there is no Hopf term at the tree level, and calculate the Hopf term induced by interactions with fermions at one loop. The possibility of generating a Hopf term via quantum effects has been studied by many groups.¹²⁻¹⁶ We give here a general, streamlined derivation which treats all dispersion relations on an equal footing, and clarifies the conditions under which a Hopf term is or is not generated. We obtain a simple answer that depends on the

low- and high-momentum behavior of the fermion dispersion relation.

The Hopf term is nonlocal when expressed in terms of the σ model field φ . However, it becomes local if we rotate φ to a fixed direction by using an $SU(2)$ matrix U : $\varphi \cdot \sigma = U^{-1} \sigma^3 U$. The Hopf term is then a Chern-Simons term for B_μ^3 , the third component of the (pure gauge) field $B_\mu = -iU \partial_\mu U^{-1}$. If φ couples to some multicomponent fermion ψ , then B_μ^3 couples minimally to the rotated fermion $\chi = U\psi$. Let us write the free Lagrangian for χ as $\mathcal{L} = \chi^\dagger S^{-1}(p)\chi$, where $S^{-1}(p)$ is the inverse of the free fermion propagator. Since in general χ has many components, S is a matrix. If S is diagonal in all indices, no Hopf term is induced. We have explicitly calculated the coefficient ϑ of the induced Hopf term for the case where S is 2×2 block diagonal. In this case the inverse propagator may be written as $S^{-1}(p) = p_0 - f - \mathbf{g} \cdot \boldsymbol{\tau}$, where $f(p_i)$ and $\mathbf{g}(p_i)$ are functions of momentum ($i = 1, 2$). For instance, in a relativistic model, g_3 is the mass, $g_{1,2}$ are the momentum components p_2 and p_1 , and $f = 0$. The Pauli matrices $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ are used as a basis (along with $\tau^0 = 1$) for the Hermitian 2×2 blocks. $\mathbf{g}$ may vary from a particular 2×2 block to another, and therefore carries a discrete index (such as isospin). The unit vector $\mathbf{g}/|\mathbf{g}|$ provides a mapping from the infinite-momentum plane into the two-sphere S_2 . Let Ω be the (oriented) fraction of total solid angle which is described by $\mathbf{g}/|\mathbf{g}|$ as $\mathbf{p} = (p_1, p_2)$ covers the whole momentum plane. The solid angle Ω also carries a diagonal index. We find that the coefficient of the induced Hopf term is

$$\vartheta = (\tfrac{1}{2} \text{ or } 1) \frac{1}{4\pi} \sum_r \int d^2\mathbf{p} \frac{1}{|\mathbf{g}|^3} \epsilon^{abc} g_a \partial_1 g_b \partial_2 g_c = \sum_r \Omega_r, \quad (1.1)$$

where the sum is taken over the diagonal indices, i.e., over all 2×2 blocks. The diagonal indices stand for

isospin, three-dimensional spin, or fermion species. If the fermion χ is self-conjugate, then the prefactor in parentheses must be $\frac{1}{2}$; otherwise, it is 1. This formula strikes with its simplicity and generality. It gives the coefficient of the induced Hopf term for many different systems without the need to repeat lengthy calculations. For systems in which the asymptotic behavior of $\mathbf{g}(p_i)$ does not depend on the direction of $\mathbf{p}$, the above formula simplifies to

$$\vartheta = \left(\frac{1}{2} \text{ or } 1\right) \frac{1}{2} \sum_r N_r \left(\left. \frac{g_3}{|\mathbf{g}|} \right|_0 - \left. \frac{g_3}{|\mathbf{g}|} \right|_\infty \right). \quad (1.2)$$

Here N is the vorticity of $\mathbf{g}$ on the momentum plane, i.e., the number of turns the projection of $\mathbf{g}$ makes in the τ^1 - τ^2 plane as one goes around the origin. This formula is readily applicable to many systems. Similar formulas for the induced fermion number and induced electromagnetic Chern-Simons term are also given, when appropriate.

An effective interaction between vacancies and the $\text{NL}\sigma$ model in quantum antiferromagnets has been calculated by Shraimann and Siggia.¹⁷ In this model the fermion propagator S is completely diagonal, and therefore no Hopf term can be induced. This is of course expected since that particular model does not explicitly violate P and T . Even if we generalize the model by introducing P - and T -violating interactions, we still find that no Hopf term is induced¹⁸.

A particularly transparent model is an isospin $\frac{1}{2}$, relativistic two-component fermion coupled to a $\text{NL}\sigma$ model by a Yukawa coupling. P and T are explicitly violated by the fermion mass term:^{19,20}

$$\mathcal{L} = \bar{\psi}(i\partial - m)\psi - g_y \bar{\psi} \boldsymbol{\varphi} \cdot \boldsymbol{\sigma} \psi, \quad (1.3)$$

where the Dirac matrices are the 2×2 Pauli matrices. The fermion field ψ carries a Dirac index and an isospin index. After the rotation of $\boldsymbol{\varphi}$, the Lagrangian is diagonal in isospin. In this model the induced Hopf term coefficient is

$$\vartheta = \begin{cases} 1, & g_y < m, \\ 0, & g_y > m. \end{cases} \quad (1.4)$$

However, in this model, the soliton statistics is not given by ϑ alone. This is because a soliton interacting with fermions may acquire a fermion number, due to vacuum polarization.²¹ Using the formalism we develop below, it is easy to find that the induced fermion number (charge) is

$$q = \begin{cases} 0, & g_y < m, \\ 1, & g_y > m. \end{cases} \quad (1.5)$$

Strictly speaking, $\vartheta = \frac{1}{2}$ and $q = \frac{1}{2}$ if $g = m$. The result (1.5) is due to Chen and Wilczek.²² The statistics parameter is obtained by adding the induced Hopf term coefficient (1.4) to the induced fermion number (1.5): $\theta = \vartheta + q = 1$. This implies that the soliton is always a fermion. We also give the corresponding results for ψ in a general isospin- j representation. The relativistic model

with two fermion families is believed to describe the flux phase of quantum antiferromagnets.²³

In the model describing the interactions of the order parameter of thin films of superfluid ^3He with fermions, P and T are violated by the formation of a p -wave condensate. We easily confirm the result of Volovik and Yakovenko¹⁶ and find that the soliton is a fermion in the absence of external fields.

It is important to point out that our result is perturbative. It remains an important question to see if a truly nonperturbative treatment can change the picture. Let us also point out that the calculations presented here can be easily extended to finite temperatures. For the models in which a Hopf term is induced with integer coefficient at zero temperature, it seems that raising the temperature gives the Hopf term a fractional coefficient, implying a violation of P and T . However, a correct interpretation of the induced terms at finite temperature requires the knowledge of the temperature dependence of all the parameters in the theory.

This paper is organized as follows. In Sec. II we review fractional statistics and show how an anyonic system can be described by an Abelian Chern-Simons theory coupled to a conserved current. We review the $\text{NL}\sigma$ model and describe why the solitons acquire fractional statistics if a topological Hopf term with noninteger coefficient is present in the action. In Sec. III we explain the general formalism. We describe the formulation of the model in terms of the curl-free non-Abelian gauge field B_μ . We discuss the symmetries of the theory. We calculate the induced Hopf term using a gradient expansion. We also calculate the induced fermion number of the soliton. In Sec. IV we look at a few specific models. Specifically, we consider the application to high- T_c superconducting materials, and find that even with certain types of P and T violation, no Hopf term is induced. We also consider a model for the A phase of superfluid ^3He films, and a relativistic fermion model, and find that a Hopf term is induced with integer coefficient. Finally, we give an example of a model in which anyons are generated with arbitrary statistics. Some specific details are given in appendixes.

II. PRELIMINARIES

A. Fractional statistics

It is by now well known that there exists a continuum of statistics in two space dimensions.^{1,2} From a geometric point of view, this fact is associated with the homotopy group of the configuration space of N identical particles. In three spatial dimensions this group is simply $\mathbf{Z}_2$, and the one-dimensional representations thereof are the trivial representation $\{1, 1\}$ and the alternating representation $\{1, -1\}$ corresponding respectively to Bose and Fermi statistics. In two dimensions this homotopy group is the braid group, whose one-dimensional unitary representations are labeled by a statistics param-

eter θ (modulo 2). Objects obeying fractional statistics ($\theta \neq 0, 1$) are colloquially called anyons.² The significance of the parameter θ lies in the phase factor $e^{i\pi\theta}$ which a many-anyon wave function will acquire when two positions are interchanged along a clockwise path that does not include any other anyon; each enclosed anyon would contribute an additional phase $e^{2i\pi\theta}$. It is possible to describe anyons using multivalued wave functions. Alternatively, they can also be described as bosons or fermions coupled to an auxiliary, *statistical* Abelian gauge field.

Fractional statistics can only occur if parity is broken; indeed, the reversal of one of the spatial coordinates would flip the exchange path of two anyons, and this flipped exchange path would produce a statistical phase factor equal to the complex conjugate of the unflipped phase factor, which is different unless θ is an integer. Likewise, time reversal must be violated. This becomes clear when an auxiliary gauge field with a Chern-Simons action is introduced, because the latter explicitly breaks these two discrete symmetries.¹⁹

The simplest and most general way of giving particles fractional statistics is to couple the conserved particle current to an Abelian gauge field with a Chern-Simons action. A generic anyon Lagrangian may be written as

$$\mathcal{L} = ej^\mu a_\mu + \frac{\alpha}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. \quad (2.1)$$

For the moment, we leave the dynamics of the current j_μ unspecified. Here the coupling e is not necessarily the electric charge. The equation of motion for the gauge field a_μ leaves no room for dynamics, as a_μ is completely determined by the current j_μ :

$$ej_\mu = -\frac{\alpha}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda. \quad (2.2)$$

We can solve the above for the auxiliary gauge field a_μ . In the Coulomb gauge, $\nabla \cdot \mathbf{a} = 0$, we obtain

$$a_i = \frac{e}{\alpha} \int d^2y \frac{\epsilon_{ij}(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^2} j_0(y), \quad (2.3)$$

$$a_0 = \frac{e}{\alpha} \int d^2y \frac{\epsilon_{ij}(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^2} j_i(y). \quad (2.4)$$

If we integrate the time component of Eq. (2.2) we obtain

$$en = \frac{\alpha}{2\pi} \int d^2x \mathbf{H} = \frac{\alpha}{2\pi} \Phi, \quad (2.5)$$

where $\mathbf{H}$ is the magnetic field of the vector potential $\mathbf{a}$, which in 2+1 dimensions is a rotational scalar, and Φ is the total flux; the total number of particles is $n = \int d^2x j_0$ and each particle is carrying a flux $2\pi e/\alpha$. If a particle of charge e completes a revolution around another identical particle carrying a flux Φ , its wave function will acquire an Aharonov-Bohm phase²⁴

$$e^{ie\Phi} = e^{2\pi i e^2/\alpha}. \quad (2.6)$$

Hence the statistics parameter of particles described by a conserved current coupled to a gauge field with a Chern-

Simons action is equal to

$$\theta = \frac{e^2}{\alpha}. \quad (2.7)$$

B. The nonlinear σ model

The conserved current of the previous subsection could be the topological current of the NL σ model. Since this model plays a central role in our discussion, we will now present a brief review thereof. The O(3) nonlinear σ model is defined by the action

$$S[\varphi] = \frac{1}{2f^2} \int d^3x \partial_\mu \varphi \cdot \partial^\mu \varphi + \vartheta H. \quad (2.8)$$

The bosonic field φ is a configuration on the surface of the two-dimensional sphere S_2 , i.e., $\varphi^2 = 1$. The second term, topological in nature, is called a Hopf term and will be described below; its role is to alter the statistics of the solitons. A static field configuration $\varphi(r, \phi)$ of finite energy must be uniform at infinity;²⁵ the coordinate space is effectively compactified to a sphere S_2 . The soliton configurations are classified by homotopy classes $\pi_2(S_2) = \mathbf{Z}$, and are labeled by an integer, the topological charge Q . It is the integral over coordinate space of the time component of the topological current j^μ given by

$$j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon^{abc} \varphi^a \partial_\nu \varphi^b \partial_\lambda \varphi^c. \quad (2.9)$$

One easily checks that the topological current is conserved, $\partial_\mu j^\mu = 0$, without making use of the equations of motion. Because of this conservation we may introduce a fictitious gauge field a_μ defined by

$$j^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda. \quad (2.10)$$

In each Q -charge sector there are minimum-energy configurations, the solitons; if we choose the vacuum configuration to be $\varphi = (0, 0, 1)$ at spatial infinity, they are given by (here $N = |Q|$)

$$\varphi = \left(\frac{2r^N}{r^{2N} + 1} \cos N\phi, \frac{2r^N}{r^{2N} + 1} \sin N\phi, \frac{r^{2N} - 1}{r^{2N} + 1} \right). \quad (2.11)$$

The mass of a soliton is given by $M = 4\pi|Q|/f^2$. If f is small, solitons are very massive and the dilute-soliton-gas approximation is a good one.

We now describe the Hopf term^{8,9}

$$H = -\pi \int d^3x j^\mu a_\mu. \quad (2.12)$$

This term is nonlocal when expressed solely in terms of the σ model field φ^a . Let us argue that the coefficient ϑ is indeed the statistics parameter for the solitons. The argument is identical to the one given in the previous subsection. We can rewrite the topological action as

$$S_H = -2\pi\vartheta \int d^3x j^\mu a_\mu + \pi\vartheta \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda . \quad (2.13)$$

This reduces to Eq. (2.12) when a_μ is eliminated through its equations of motion. Let us concentrate on solitons with unit topological charge $Q = 1$. Integrating the time component of the equations of motion shows that a unit (1) magnetic flux is carried by each soliton. If we compare with Eqs. (2.6) and (2.1) we see that the Aharonov-Bohm phase produced by taking one soliton around another is $\exp(2\pi i\vartheta)$. Hence, the solitons with $Q = 1$ obey fractional statistics with parameter $\theta = \vartheta$.

A geometric argument for the statistics and the spin of these solitons can be given using the notion of linking numbers.⁹ Note that the Hopf term H is actually invariant under any smooth variation of the field φ^a , and that H/π is the index of mappings from the three-dimensional sphere S_3 to the configuration space S_2 . The homotopy group $\pi_3(S_2) = \mathbb{Z}$, and H/π is restricted to these values.⁹

The spin-statistics theorem holds true in this model. This means that the spin of the soliton is $\vartheta/2$. This can be easily seen as follows. Let us adiabatically rotate the soliton of charge Q by quantizing the rotational collective coordinate:^{26,27} $\phi \mapsto \phi + \alpha(t)$. We can easily calculate the Hopf term associated with this rotation:²⁸

$$H = \frac{1}{2}Q^2 \int dt \dot{\alpha} . \quad (2.14)$$

Altogether, the semiclassical, effective soliton action for the rotational collective coordinate α is

$$S[\alpha] = \int dt \left(-M + \frac{1}{2}I\dot{\alpha}^2 + \frac{1}{2}\vartheta Q^2 \dot{\alpha} \right) . \quad (2.15)$$

M is the soliton mass and I its moment of inertia. After a complete adiabatic rotation by 2π , the Hopf part of the effective action is just $\pi\vartheta Q^2$. The phase factor produced by the adiabatic rotation is then $e^{iS} = e^{2\pi iJ} = e^{i\pi\vartheta Q^2}$, where S is the rotational part of the action in the adiabatic limit. Hence the spin of the soliton of charge Q is $J = \frac{1}{2}\vartheta Q^2$. The total angular momentum is obtained by adding the canonical piece J_c . The statistics of a charge Q soliton is $Q^2\vartheta$. The expression for the total angular momentum of the soliton, $J = J_c + \frac{1}{2}Q^2\vartheta$, is exactly the one dictated by the Poincaré symmetry of the theory. Note that in the absence of the Hopf term ($\vartheta = 0$) the angular momentum of the soliton is identified with the third component of the isospin.

III. GENERAL FORMALISM

In this section we introduce fermions interacting with the $NL\sigma$ model. Under the broad assumption that the interactions obey the $O(3)$ symmetry of the $NL\sigma$ model, the system is represented in terms of a rotated fermion and a curl-free non-Abelian gauge field minimally coupled to it. In this language the Hopf term is nothing but a Chern-Simons term. We proceed to calculate the one-loop-induced Chern-Simons term and obtain a gen-

eral answer that depends only on the low- and high-momentum behavior of the fermion propagator. We also calculate the induced fermion number of the $NL\sigma$ model solitons. The results derived in this section are quite general.

In two spatial dimensions the rotation group is the Abelian group $U(1)$. Therefore one can write rotation invariant Lagrangians with a single complex fermion ψ . If we also require Lorentz invariance, two complex fermions are necessary, for the particle and its antiparticle respectively: a Dirac fermion in $2+1$ dimensions has two complex components.

We require that interactions between fermions and the $NL\sigma$ model be invariant under the full three-dimensional rotation symmetry group $SU(2)$. From the point of view of two-dimensional fermions, the three-dimensional rotation symmetry acts like an internal isospin symmetry. Since φ is an isospin-1 field, the fermions must live in an isospin- $\frac{1}{2}$ (or higher) representation of $SU(2)$. For the moment we consider isospin- $\frac{1}{2}$ fermions and in Appendix D we indicate in what ways the results change for higher isospin.

We find it convenient to reformulate the $NL\sigma$ model in terms of a curl-free non-Abelian gauge field. We illustrate this by considering the simple example of an isospin- $\frac{1}{2}$ Dirac fermion ψ interacting with the $NL\sigma$ via a Yukawa coupling:

$$\mathcal{L}_f = \bar{\psi}(i\cancel{D} - m)\psi - g_y \bar{\psi}\varphi\psi , \quad (3.1)$$

where the matrix field $\varphi = \varphi^a \sigma^a$. The gamma matrices are $\gamma^0 = \tau^3$, $\gamma^1 = i\tau^1$, $\gamma^2 = i\tau^2$, and both σ^a and τ^a are Pauli matrices. The field ψ has four complex components. Under $SU(2)$ transformations, the above fields transform as follows: $\psi \mapsto U\psi$ and $\varphi \mapsto U\varphi U^{-1}$, where U is an $SU(2)$ matrix. The above model violates parity because of the presence of a Dirac mass term.²⁰

Now, let us define

$$\varphi = U^{-1}\sigma^3 U, \quad \chi = U\psi, \quad B_\mu = -iU\partial_\mu U^{-1} , \quad (3.2)$$

where the matrix $U(x) \in SU(2)$ rotates the scalar field φ into the third axis of isospin space. The field χ is a rotated fermion field and $B_\mu = B_\mu^a \sigma^a$ is the curl-free non-Abelian gauge field corresponding to U . The solitons are now magnetic monopole configurations of the gauge field B_μ . In terms of the new fields the Lagrangian (3.1) is

$$\mathcal{L} = \bar{\chi}(i\cancel{D} - m - g_y \sigma^3 - \cancel{B})\chi . \quad (3.3)$$

The rotated fermion is coupled minimally to B_μ and the fermion kinetic term is diagonal in isospin space because it only involves the matrix σ^3 . This is a general feature of $SU(2)$ -invariant fermion interactions. We will illustrate this with more examples in the next section. From here on we will consider a general rotated fermion χ with propagator $S(p)$ (possibly containing σ^3) that couples minimally to B_μ . The Lagrangian has the general form

$$\mathcal{L}_f = \chi^\dagger S^{-1}(p - B)\chi, \quad (3.4)$$

where $S(p)$ is the propagator of χ . S is diagonal in isospin indices, but not necessarily diagonal in others, such as Dirac indices or family indices. For the model (3.1), the inverse propagator is $S^{-1}(p) = p_0 - \tau^3(m + g\sigma^3) - p_1\tau^2 + p_2\tau^1$.

A. Symmetries

Before going further, let us see how to express the soliton current and the NL σ action in terms of the gauge field B_μ . The topological current can be written as

$$j^\mu = \frac{1}{32\pi i} \epsilon^{\mu\nu\lambda} \text{tr}(\varphi[\partial_\nu\varphi, \partial_\lambda\varphi]). \quad (3.5)$$

Since $\varphi = U^{-1}\sigma^3 U$ it readily follows that $\partial_\mu\varphi = iU^{-1}[B_\mu, \sigma^3]U$. The above current may then be rewritten as $j^\mu = (1/2\pi)\epsilon^{\mu\nu\lambda}\epsilon^{abc}B_\nu^a B_\lambda^b$. The curl-free condition for B_μ ($F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu] = 0$) can be cast in the useful form

$$\epsilon^{\mu\nu\lambda}\epsilon^{abc}B_\nu^a B_\lambda^b = \epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda^c. \quad (3.6)$$

Using (3.6) the topological current (3.5) becomes

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu B_\lambda^3. \quad (3.7)$$

Comparing (3.7) with the defining equation, $j^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$, for the Abelian, statistics gauge field a_μ , we find that (up to an Abelian gauge transformation)²⁹

$$a_\mu = \frac{1}{2\pi} B_\mu^3. \quad (3.8)$$

The topological action is given by

$$S_H = -\pi\theta \int d^3x j^\mu a_\mu = \frac{-\theta}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} B_\mu^3 \partial_\nu B_\lambda^3. \quad (3.9)$$

It can be interpreted as a Chern-Simons term for B_μ^3 . In terms of the fictitious gauge field B_μ , the kinetic energy term of the σ model is a mass term for B_μ^1 and B_μ^2 :

$$\frac{1}{2f^2} \partial_\mu \varphi^a \partial^\mu \varphi^a = \frac{2}{f^2} (B_\mu^1 B^{1\mu} + B_\mu^2 B^{2\mu}). \quad (3.10)$$

Let us point out that no nontrivial Jacobian arises when going from the φ, ψ set of variables to the B_μ^a, χ set of variables.³⁰

In terms of the field φ , the NL σ model action has a global SU(2) symmetry. Let α be its infinitesimal generator. The NL σ model Lagrangian is invariant with respect to variations $\delta_\alpha\varphi = i[\alpha, \varphi]$. Let us find the realization of this global SU(2) symmetry in terms of the field B_μ^a . Since $\varphi = U^{-1}\sigma^3 U$, we see that $\delta\varphi = [\delta U^{-1}U, \varphi]$. The general solution for the variation of U is $\delta U = -iU\alpha + i\beta U\varphi$, where $\beta(x)$ is some infinitesimal function. It arises because of a local phase ambiguity in U : Let $U'(1)$ be the subgroup of SU(2) generated by σ^3 , and let $V(x) = \exp(i\beta\sigma^3/2) \in U'(1)$; then the matrix VU

also rotates φ into σ^3 and is equivalent to U . The gauge field $B_\mu = -iU\partial_\mu U^{-1}$ transforms as

$$\delta B_\mu = -\sigma^3 \partial_\mu \beta - i\beta[B_\mu, \sigma^3]. \quad (3.11)$$

In components, this translates into $\delta B_\mu^1 = 2\beta B_\mu^2$, $\delta B_\mu^2 = -2\beta B_\mu^1$, $\delta B_\mu^3 = -\partial_\mu \beta$. These transformations correspond to a local Abelian gauge symmetry in which B_μ^3 plays the role of the gauge field and in which χ and B^1, B^2 are simply rotated.³¹ Obviously, the NL σ model action in terms of B_μ (as written above) obeys this symmetry. It might seem surprising that, starting from a theory without local symmetry, we end up with a theory with Abelian gauge invariance just by a change of variables. However, our change of variables apparently increases the number of degrees of freedom. The extra local symmetry emerges precisely to keep the number of degrees of freedom the same.

The effective action obtained by integrating out the fermions should also have the same symmetry. Only two distinct terms can be written that satisfy the above Abelian gauge symmetry [independent of the curl-free condition (3.6)] and that contribute to the topological action (3.7):

$$C_1 \epsilon^{\mu\nu\lambda} B_\mu^3 \partial_\nu B_\lambda^3 + C_2 \epsilon^{\mu\nu\lambda} \left(\sum_{a \neq 3} B_\mu^a \partial_\nu B_\lambda^a - 2\epsilon^{ab3} B_\mu^a B_\nu^b B_\lambda^3 \right). \quad (3.12)$$

It is important to stress that the coefficients C_1 and C_2 may be nonzero only if P and T are violated. However, even if P and T are violated, it is still possible that C_1 and C_2 vanish. The second term vanishes upon using the curl-free relation (3.6) for B_μ . Therefore we can restrict our calculation of the effective action to terms of the form $\epsilon^{\mu\nu\lambda} B_\mu^3 \partial_\nu B_\lambda^3$. This argument would fail if this gauge symmetry were anomalous. However, a general argument, given in Appendix A, shows that the gauge symmetry is preserved at the one-loop level. Furthermore, it can be shown that there are no higher-loop corrections to the one-loop evaluation of (3.12).³² The above is also true even if B_μ^3 and $B_\mu^{1,2}$ do not couple with the same strength to the fermions. This may happen if derivative couplings are present, and we will give an example in the next section. This is basically due to the fact that $B_\mu^{1,2}$ are not the gauge fields of the $U'(1)$ symmetry, but instead behave like matter fields of $U'(1)$.

It may be puzzling that the terms in (3.12) have manifest Lorentz invariance, even though they may be induced by fermions with nonrelativistic dispersion relations. However, this Lorentz invariance is only a consequence of the gauge invariance for the Chern-Simons term. Specifically, if only rotational invariance was required, the first term of (3.12) would instead be $a_1 \epsilon^{ij} B_0^3 \partial_i B_j^3 - a_2 \epsilon^{ij} B_i^3 \partial_0 B_j^3$. The requirement of gauge invariance enforces the condition $a_1 = 2a_2$, which guarantees Lorentz invariance. A similar remark applies for the second term of (3.12).

B. The induced Hopf term

In this subsection we derive the expression for the coefficient of the induced Hopf term. Let us define the vertex $V(p, B) = S^{-1}(p) - S^{-1}(p - B)$. The one-loop effective action is

$$S_{\text{eff}}[B] = -i \text{Tr} \ln[1 - S(p)V(p, B)] \\ = i \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[S(p)V(p, B)]^n. \quad (3.13)$$

If the fermion χ is self-conjugate, the path integral yields the square-root of the fermionic determinant, and a factor of $\frac{1}{2}$ must appear in front of the above expression. This factor will be suppressed for now, only to be reintroduced later for the relevant examples. The symbol Tr stands for $\int [d^3p/(2\pi)^3] \int d^3x \text{tr}$, where tr is a trace over discrete indices. We have shown in the last subsection that only the second term in this series will contribute to the Hopf term. It is sufficient to keep in $V(p, B)$ only the terms linear in B ; higher powers will contain B_μ contracted in ways that do not occur in the Chern-Simons term $\epsilon^{\mu\nu\lambda} B_\mu^3 \partial_\nu B_\lambda^3$. Therefore we may write

$$V(p, B) = \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\mu}, B_\mu^a \sigma^a \right\} + \dots \quad (3.14)$$

The quadratic piece ($n = 2$) of the effective action is

$$S_{\text{eff}}^{(2)}[B] \\ = \frac{i}{2} \text{Tr} \left(S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\mu}, B_\mu^a \sigma^a \right\} S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\lambda}, B_\lambda^b \sigma^b \right\} \right). \quad (3.15)$$

By using a gradient expansion³³ we can easily extract the Chern-Simons term from (3.15):

$$S_{\text{CS}}[B] = \frac{1}{2} \text{Tr} \left(S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right) \int d^3x B_\mu^3 \partial_\nu B_\lambda^3. \quad (3.16)$$

Exploiting the cyclic property of the trace and the antisymmetry of the spatial integral, we rewrite the Eq. (3.16) in the form of the topological action (3.9), with

$$\vartheta = \frac{-1}{24\pi^2} \epsilon^{\mu\nu\lambda} \int d^3p \text{tr} \left(S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right). \quad (3.17)$$

The remaining trace is taken over isospin space and other discrete indices; S is already diagonal in isospin space. At this stage we can already conclude that $\vartheta = 0$ if S is diagonal in all indices, or if it can be diagonalized with a momentum-independent matrix, even though there may be P - and T -violating interactions. Indeed, in that case all the factors within the trace commute, making the trace itself symmetric in $(\mu\nu\lambda)$, while multiplied by $\epsilon^{\mu\nu\lambda}$. Equation (3.17) resembles the Wess-Zumino term, except

that the matrix S is Hermitian, not unitary. Nevertheless, we can show that the integrand is a total derivative (see Appendix B).

Let us calculate ϑ for the case in which $S(p)$ is 2×2 block diagonal. Let us parametrize each block with the Pauli matrices τ . This τ space is like a Dirac spinor space. Then the inverse propagator can be written as

$$S^{-1}(p) = p_0 + f(p_i) - g(p_i) \cdot \tau. \quad (3.18)$$

Dependence on p_0 is confined to the first term, and $f(p_i)$ is assumed to be a bounded function of momentum (this restriction is trivially met in all the systems we have looked at). Both $f(p_i)$ and $g(p_i)$ may contain σ^3 or some family index.

By using the cyclic property of the trace and the fact that $\partial_0 S^{-1} = 1$, we write ϑ as

$$\vartheta = \frac{-1}{8\pi^2} \int d^3p \text{tr} \left(S \left[S \frac{\partial S^{-1}}{\partial p_1}, S \frac{\partial S^{-1}}{\partial p_2} \right] \right). \quad (3.19)$$

The trace is taken over τ space and over other, diagonal indices which we collectively call r . Given the above form for S^{-1} , we have $S = (p_0 + f + g \cdot \tau)/D$, with $D = (p_0 + f)^2 - |g|^2$. It is a straightforward exercise to calculate the trace over τ space and to find that

$$\vartheta = \frac{-i}{2\pi^2} \sum_r \int d^3p \frac{1}{D^2} \epsilon^{abc} g_a \partial_1 g_b \partial_2 g_c. \quad (3.20)$$

It follows immediately that $\vartheta = 0$ if g_1 and g_2 are momentum independent. The next step is to integrate over p_0 . Since f is bounded, we may shift $p_0 \rightarrow p_0 - f$, and the integration (after a Wick rotation) gives

$$\vartheta = \frac{1}{4\pi} \sum_r \int d^2\mathbf{p} \frac{1}{|\mathbf{g}|^3} \epsilon^{abc} g_a \partial_1 g_b \partial_2 g_c. \quad (3.21)$$

The integrand is nothing but the Jacobian from the momentum plane to the unit sphere $\mathbf{S}_2$ described by the unit vector $\mathbf{g}/|\mathbf{g}|$ (see Appendix C). Let Ω be the fraction of solid angle in $\mathbf{S}_2$ which is covered by this unit vector as (p_1, p_2) covers the whole momentum plane. Then we have the simple result

$$\vartheta = \left(\frac{1}{2} \text{ or } 1\right) \sum_r \Omega_r. \quad (3.22)$$

The factor in parentheses is $\frac{1}{2}$ if χ is self-conjugate, and 1 otherwise. This is a remarkably simple and general result.

With a few mild assumptions on the behavior of $\mathbf{g}$, we can make this formula more explicit. Let us assume that the ratio $g_3/|\mathbf{g}|$, which is the cosine of the angle that $\mathbf{g}$ makes with the τ^3 direction, is independent of the direction of momentum at zero and infinite momentum. Let N be the vorticity of $\mathbf{g}$ on the momentum plane, i.e., the number of turns the projection of $\mathbf{g}$ makes in the τ^1 - τ^2 plane as one goes around the origin on the momentum plane. (The vorticity is zero if the propagator is diagonal in τ space, or if g_1 and g_2 are momentum

independent.) Then, the coefficient of the induced Hopf term is

$$\vartheta = (\tfrac{1}{2} \text{ or } 1) \tfrac{1}{2} \sum_r \left[N_r \left(\left. \frac{g_3}{|g|} \right|_0 - \left. \frac{g_3}{|g|} \right|_\infty \right) \right]. \quad (3.23)$$

The sum is again taken over diagonal indices. From this formula it becomes clear that ϑ is almost never a continuous function of the model's parameters: For $g_3(0)/|g(0)|$ or $g_3(\infty)/|g(\infty)|$ not to be integers, there must be fine-tuning in the power laws that govern g . Specifically, $g_3(p_i)$ must obey the same power law as $g_{1,2}(p_i)$ as $p^2 \rightarrow 0$ or as $p^2 \rightarrow \infty$.

Let us mention now in what circumstances self-conjugate fermions might be used. Let $\psi = (\psi_1, \psi_2)$ be a complex spin doublet, where the two components have opposite planar spins. A term such as $a\psi_1\psi_2 + a^*\psi_2^*\psi_1^*$, while rotation invariant and real, cannot be put in the form (3.4) right away. In order to do so we introduce a four-component, self-conjugate fermion $\Psi = (\psi, i\sigma^2\psi^*)$. Notice that ψ and $i\sigma^2\psi^*$ transform the same way under $SU(2)$ rotations, since $\sigma^2 U \sigma^2 = U^*$, where $U \in SU(2)$. The term $a\psi_1\psi_2 + a^*\psi_2^*\psi_1^*$ can now be written as $\tfrac{1}{2}\Psi^\dagger \mathbf{a} \cdot \boldsymbol{\tau} \Psi$, with $\mathbf{a} = (a_1, a_2)$ and $a = a_1 + ia_2$. For a nonrelativistic theory, nondiagonal pieces correspond to BCS-type condensation with an electrically charged condensate.

Finally, note that the whole formalism may be easily extended to finite temperatures.³⁴

C. Fermion number of the soliton

In this subsection we derive the formula for the induced fermion number (and electromagnetic charge, for charged fermions) of the soliton. The present derivation is relevant to systems in which the fermion number symmetry is not spontaneously broken by charged condensate. For a nonrelativistic system this occurs when the condensate is neutral, which implies that S is diagonal. We will show that in such a case the soliton charge is not induced. This means that the following discussion is relevant to relativistic systems only. Let us introduce an external electromagnetic field A_μ that couples minimally to the fermion ψ (or, equivalently, to χ) and equally to all isospin components. The effective action may include a coupling between A_μ and the topological current $(1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda^3$, in which case the soliton acquires an induced charge (A_μ may be considered merely as a source term for the charge). This amounts to computing the expectation value of the fermion current in a soliton background, $\langle \bar{\psi}\gamma^\mu\psi \rangle = \text{Tr}\gamma^\mu S_F$.

The result can be extracted from the calculation of the preceding subsection. Let us assume that we have added an Abelian component $B_\mu^0 = A_\mu$ with generator $\sigma^0 = 1$ to the non-Abelian field B_μ . The term that represents a soliton electromagnetic coupling is

$$\frac{q}{2\pi} \int d^3x \epsilon^{\mu\nu\lambda} B_\mu^0 \partial_\nu B_\lambda^3 = q \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda, \quad (3.24)$$

and one can show that the other term allowed by symmetry ($\epsilon^{\mu\nu\lambda} B_\mu^0 B_\nu^1 B_\lambda^2$) is not generated. The term in Eq. (3.24) will arise in the effective action as

$$-\tfrac{1}{2}\text{Tr} \left(\sigma^3 S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right) \times \int d^3x (B_\mu^0 \partial_\nu B_\lambda^3 + B_\mu^3 \partial_\nu B_\lambda^0). \quad (3.25)$$

It follows that the induced soliton charge is

$$q = \frac{\pi}{3} \text{Tr} \left(\sigma^3 S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right) = \tfrac{1}{2} \sum_r \sigma_{rr}^3 \Omega_r. \quad (3.26)$$

The induced charge is therefore zero if the Lagrangian is diagonal in τ space, just like the Hopf term coefficient. An example will be given in the next section.

Using the same method we can also compute the induced electromagnetic Chern-Simons term $(\alpha/4\pi)\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$. The coefficient α is given by

$$\alpha = \frac{\pi}{3} \text{Tr} \left(S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right) = \tfrac{1}{2} \sum_r \Omega_r. \quad (3.27)$$

Therefore α is the same as the coefficient ϑ of the induced Hopf term. We stress that these results are valid only if all the components of the fermion χ transform the same way under electromagnetic gauge transformations.

IV. EXAMPLES

A. A relativistic model

As a first example of application of the general result, we consider a generalization of the relativistic model (3.1):

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g_y \bar{\psi}\varphi\psi - h\bar{\psi}\partial\!\!\!/\varphi\psi + ik\bar{\psi}[\varphi, \partial\!\!\!/\varphi]\psi. \quad (4.1)$$

A two-family version of this model may also be relevant to high- T_c superconductivity, because it is believed to describe the flux phase of quantum antiferromagnets. In the model (4.1) we have included two types of derivative couplings. The purpose of this generalization is to show that the derivative couplings have no effect on the result, even though they may be an additional source of P and T violation.³⁵ The Dirac mass term breaks P and T . Even if $m = 0$, P and T are still broken if the Yukawa coupling g and the derivative coupling h are simultaneously nonzero.

Let us rotate φ to σ^3 , remembering that $\partial_\mu\varphi = iU^{-1}[B_\mu, \sigma^3]U$. We obtain

$$\mathcal{L} = \bar{\chi} \left(i\partial\!\!\!/ - m - g_y \sigma^3 - \not{B} + 2h \sum_{a,b \neq 3} \not{B}^a \epsilon^{ab} \sigma^b + 4k \sum_{a \neq 3} \not{B}^a \sigma^a \right) \chi. \quad (4.2)$$

By defining the rotated generators $\tilde{\sigma}^1 = \eta^{-1}\{(1+4k)\sigma^1 + 2h\sigma^2\}$ and $\tilde{\sigma}^2 = \eta^{-1}\{(1+4k)\sigma^2 - 2h\sigma^1\}$, where $\eta = \sqrt{(1+4k)^2 + 4h^2}$, we can rewrite the rotated Lagrangian as

$$\mathcal{L} = \bar{\chi} \left(i\tilde{\not{D}} - m - g_y \sigma^3 - \not{B}^3 \sigma^3 - \eta \sum_{a=1,2} \not{B}^a \sigma^a \right) \chi. \quad (4.3)$$

The new generators $\sigma^3, \tilde{\sigma}^1, \tilde{\sigma}^2$ obey the same algebra as the Pauli matrices. They are merely rotations thereof. The coupling of χ to B_μ is minimal. The only effect of the derivative couplings is to modify the strength of the couplings to B_μ^1 and B_μ^2 relative to B_μ^3 . This is consistent with the general symmetry (3.11). The general argument of the last section tells us that the components $B_\mu^{1,2}$ do not contribute to the induced Hopf term, and therefore we may ignore them. Despite the fact that the derivative couplings h and k are a source of P and T violation, they are of no consequence. This is a particular example of the fact that P and T violation are necessary but not sufficient conditions for the generation of a Hopf term.

The inverse propagator for the present model is $S^{-1}(p) = \gamma^0(\not{p} - M)$, with $M = m + g\sigma^3$. It is of the form (3.18) with $\mathbf{g} = (p_2, -p_1, M)$ and $|\mathbf{g}|^2 = p^2 + M^2$. In particular, $\mathbf{g}_r(0)/|\mathbf{g}_r(0)| = (0, 0, \text{sgn}(M_r))$ and $\mathbf{g}(\infty)$ lies on the τ^1 - τ^2 plane. The formula (3.23) may be applied. One easily sees that the vorticity N_r of $\mathbf{g}$ is 1, and the coefficient of the induced Hopf term is

$$\vartheta = \frac{1}{2} \sum_r \text{sgn}(M_r) = \begin{cases} 1, & g_y > m, \\ 0, & g_y < m. \end{cases} \quad (4.4)$$

There is another contribution to the spin and statistics of the solitons, from the induced fermion number. From Eq. (4.5), we find that the induced fermion number (or electromagnetic charge) is

$$q = \frac{1}{2} \sum_r [\sigma_{rr}^3 \text{sgn}(M_r)] = \begin{cases} 0, & g_y > m, \\ 1, & g_y < m. \end{cases} \quad (4.5)$$

Note that $\vartheta = \frac{1}{2}$ and $q = \frac{1}{2}$ if $g_y = m$. The results (4.4) and (4.5) together determine the statistics of the solitons. The statistics parameter is $\theta = q + \vartheta = 1$, irrespective of the coupling. This result complements the result of Chen and Wilczek,²² where (4.5) was computed. The coefficient α of the induced electromagnetic Chern-Simons term is the same as ϑ , since all the components of ψ transform the same way under an electromagnetic gauge transformation. If two independent families of fermions are present, as it may be in the flux phase of quantum antiferromagnets, then the results (4.4) and (4.5) are multiplied by a factor of 2. In that case the soliton remains a boson, but the z quanta, in the deconfined phase of the CP_1 formulation, are semions.

Let us point out that it is possible to add to the Lagrangian (4.1) a nonminimal coupling to B_μ^3 : $\delta\mathcal{L} = c\bar{\psi}\gamma^\mu\psi j_\mu$, where j_μ is the topological current (3.5). *A priori* a term like $j^\mu B_\mu^3$ (which is nothing but a Hopf

term) might be induced in the effective action by this new interaction, but a simple calculation shows that this is not the case.

B. Models of quantum antiferromagnets

In the theory of two-dimensional quantum antiferromagnets, the $NL\sigma$ model describes the long-wavelength limit of spin waves. It is believed that the physics of high- T_c materials is captured by the model of a two-dimensional quantum antiferromagnet doped with vacancies. At low doping, in the long-wavelength limit, this model can be approximated by the $NL\sigma$ model coupled to fermionic holes. The hole ground state occurs at the center of the face of the reduced Brillouin zone. Since the Brillouin zone contains two inequivalent minima, in the long-wavelength limit there are two fermion species. It is argued by some that the hole ground state is not P and T invariant, and therefore the system might exhibit anyonic excitations. We can model this by explicitly introducing P - and T -violating interactions. In the framework of the t - J model, a simple picture of the dynamics of a hole in the antiferromagnetic background has been developed by Shraiman and Siggia.¹⁷ In their picture the long-ranged effect of the hole is a dipolar distortion of the background antiferromagnetism. *A priori*, at the level of this model, both plane waves solutions and soliton solutions are expected to be of equal importance. Therefore, if P and T are violated, one may expect to find anyons. The effective Hamiltonian of Shraiman and Siggia¹⁷ does not violate P and T , and, in the continuum limit, it is given by

$$\begin{aligned} \mathcal{H} = & \psi^\dagger \epsilon(p_i + i\frac{1}{2}[\varphi, \nabla_i \varphi]) \psi \\ & - i h_1 \psi^\dagger [(p_i[\varphi, \nabla_i \varphi] + [\varphi, \nabla_i \varphi] p_i)] \psi \\ & - \psi^\dagger (i h_2 [\varphi, \dot{\varphi}] + i h_3 n_i [\varphi, \nabla_i \varphi]) \psi \\ & - \psi^\dagger (h_5 \partial_i \varphi^a \partial_i \varphi^a + h_4 \dot{\varphi}^a \dot{\varphi}^a) \psi, \end{aligned} \quad (4.6)$$

where $\epsilon(p_i)$ is the kinetic energy, quadratic in momentum but anisotropic. The fermion field ψ describes the hole. It is a two-component spinor, and the spin structure is inherited from the two sublattices of the antiferromagnet. The $NL\sigma$ -model field φ describes the staggered magnetization. The sum over fermion species is implicit. The vector $\mathbf{n}$ is the phase velocity of the spin wave, and is of order 1 in the ground state. The coupling h_1 describes the interaction between the spin current and the magnetization current. The coupling h_2 , negligible near the ground state, describes the interaction between the spin density and the magnetization density. The coupling h_3 , the most important near the ground state, describes the interaction of the spin density with the magnetization density induced by the phase velocity of the spin wave. Since this model does not violate P and T , no Hopf term can be induced, nor is any electromagnetic Chern-Simons term. We now want to model the P and T noninvariance of the ground state by adding to the above Hamiltonian

P - and T -violating interactions.¹⁸ For instance, we could add the pieces

$$\delta\mathcal{H} = h_6 \epsilon^{ij} (\psi^\dagger \nabla_i \varphi \nabla_j \psi + \text{H.c.}) + h_7 \psi^\dagger \dot{\varphi} \psi + h_8 \psi^\dagger \varphi \psi . \quad (4.7)$$

The coupling h_6 describes the interaction between the spin current and the gradient of the staggered magnetization, and is P violating. The interaction of the spin density to the slow time variation of the staggered magnetization is described by the coupling h_7 , which is T violating. In addition, we have introduced a Yukawa coupling h_8 . Even after introducing the P - and T -violating interactions (4.7), the total Hamiltonian $\mathcal{H} + \delta\mathcal{H}$ remains diagonal after the rotation $\varphi \rightarrow \sigma^3$, i.e., $\mathbf{g} \cdot \boldsymbol{\tau} = g_3(\mathbf{p}_i) \tau^3$. Therefore, based on the general arguments of Sec. III, no Hopf term is induced. This is another example of the fact that P and T violation is not sufficient for anyons to appear in the spectrum.

C. Models of superfluid ^3He films

The NL σ model is also used to describe the p -wave condensate in films of superfluid ^3He in A -type phases.^{7,16} A starting point for the description of the superfluid ^3He in the A phase is the following four-Fermi effective Hamiltonian:

$$\mathcal{H} = \psi^\dagger \epsilon(\mathbf{p}) \psi + \frac{1}{2} h [\psi_\alpha^* (-i\vec{\nabla}) \psi_\beta^*] [\psi_\beta (-i\vec{\nabla}) \psi_\alpha] , \quad (4.8)$$

where the fermion field ψ describes ^3He atoms. The dispersion relation is $\epsilon(\mathbf{p}) = \mathbf{p}^2/2m - \mu$, where μ is the chemical potential (at zero temperature, it is the Fermi energy). The four-Fermi interaction describes a p -wave pairing. The order parameter³⁶ can be written in a matrix form:

$$\mathbf{R} = \mathbf{R}^a \sigma^a = \frac{1}{4} h [-\psi^\dagger i \sigma^2 \sigma^a (-i\vec{\nabla}) \psi] \sigma^a . \quad (4.9)$$

Using the above order parameter, the Hamiltonian may be linearized:³⁷

$$\mathcal{H} = \psi^\dagger \epsilon(\mathbf{p}) \psi + \frac{1}{4} [\psi^\dagger (\mathbf{R} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{R}) i \sigma^2 \psi^*] + \text{c.c.} \quad (4.10)$$

From the Hamiltonian (4.10) one can easily derive the gap equation for the order parameter $\mathbf{R}$. The theory has a $U(1)$ gauge invariance, under which

$$\delta\psi = i\alpha\psi, \quad \delta\mathbf{R} = 2i\alpha\mathbf{R} . \quad (4.11)$$

This gauge invariance is spontaneously broken by the condensate. It is important to point out that the interaction term is gauge invariant without the introduction of the gauge field A_μ . In other words, minimal coupling needs only be applied to the kinetic term. In A -type phases, the order parameter can be factorized into spin and orbital degrees of freedom: $\mathbf{R} = c\varphi\boldsymbol{\Delta}$, where the spin degree of freedom φ is the NL σ model field and the orbital degree of freedom is described by a complex vector $\boldsymbol{\Delta}$. In

the superfluid phase of thin films, $\boldsymbol{\Delta}$ lies in the plane of the film and can be parametrized as $\boldsymbol{\Delta} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \exp(i\eta)$. The phase $\exp(i\eta)$ is analogous to the usual BCS condensate. The orbital angular momentum is perpendicular to the plane of the film. The Hamiltonian (4.10) contains terms of the form $\psi \mathbf{p} \psi$ and $\psi^* \mathbf{p} \psi^*$, and therefore it is convenient to use self-conjugate fermions $\Psi = (\psi, i\sigma^2 \psi^*)$. In this four-component formalism, the Lagrangian of the system takes the form

$$\mathcal{L} = \Psi^\dagger [p_0 - \epsilon(\mathbf{p}) - \frac{1}{2} c \{\mathbf{p} \cdot \boldsymbol{\tau}, \varphi\}] \Psi \quad (4.12)$$

$$+ c \Psi^\dagger [\{\mathbf{p} \cdot \boldsymbol{\tau}, \varphi \sin^2(\eta/2)\}] \Psi \\ - \frac{1}{2} c \Psi^\dagger [\{\mathbf{p} \cdot (\hat{\mathbf{z}} \times \boldsymbol{\tau}), \varphi \sin \eta\}] \Psi . \quad (4.13)$$

Here the vector $\boldsymbol{\tau}$ stands for (τ^1, τ^2) only. Notice that P and T are violated in this model, in the $\mathbf{p} \cdot \boldsymbol{\tau}$ piece.

We can pick a gauge for the $U(1)$ symmetry by setting $\eta = 0$, after which only the first line of (4.13) survives.

The rotation $\varphi \rightarrow \sigma^3$ brings the Lagrangian into the general form (3.4), with the inverse propagator

$$S^{-1}(\mathbf{p}) = p_0 - \tau^3 \epsilon(\mathbf{p}) + c \sigma^3 (\mathbf{p} \cdot \boldsymbol{\tau}) . \quad (4.14)$$

The vector $\mathbf{g}$ is equal to $(c\sigma^3 p_1, c\sigma^3 p_2, \epsilon)$. At zero momentum, $\mathbf{g}$ points in the τ^3 direction and $g_3(0)/|g(0)| = \text{sgn}(\mu)$. As $\mathbf{p} \rightarrow \infty$, $\mathbf{g}$ also points in the positive τ^3 direction, and $g_3(\infty)/|g(\infty)| = 1$. The vorticity N of $\mathbf{g}$ is 1, and therefore the Hopf-term coefficient is

$$\vartheta = 1 \quad (\mu > 0) . \quad (4.15)$$

If $\mu < 0$, $\vartheta = 0$; however, this case is unphysical at zero temperature. At finite temperatures, the negative chemical potential indicates that we are above the critical temperature. We have performed the sum over spins, and multiplied an extra factor of $\frac{1}{2}$ since Ψ is self-conjugate.

It is also possible to calculate the induced charge of the soliton. Minimal coupling in the four-component formalism is implemented by the substitution $p_\mu \rightarrow p_\mu - A_\mu \tau^3$ in the kinetic term. We reiterate that minimal coupling need not be implemented on the interaction term since it is already gauge invariant. The field A_μ need not be the actual electromagnetic field, but simply the source term for the fermion current. Therefore we have to modify the calculation outlined in Sec. III C, which was not geared towards systems with self-conjugate fermions. The choice of the gauge $\eta = 0$ implies that the effective action obtained by integrating out Ψ will not be manifestly gauge invariant in the electromagnetic $U(1)$. However, it is not difficult to extract the soliton charge from it, nor to later restore the $U(1)$ gauge invariance. After using a gradient expansion, the term in the effective action that involves a coupling between the soliton current and the electromagnetic field is

$$-\frac{1}{2} \text{Tr} \left(S \frac{\partial S_0^{-1}}{\partial p_\mu} \tau^3 S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right) \int d^3x A_\mu \partial_\nu B_\lambda^3 , \quad (4.16)$$

where $S_0^{-1} = p_0 - \epsilon(\mathbf{p})\tau^3$. Explicit calculation of the traces and integrals shows that the expression (4.16) vanishes. Therefore, the statistics of the soliton is given by the Hopf-term coefficient only.

By the same method we may also calculate the induced electromagnetic Chern-Simons term. In the ^3He system, the coefficient of the electromagnetic Chern-Simons term is the Hall conductance. Since gauge invariance is not explicit, the induced Chern-Simons term has the general form

$$\mathcal{L}_{CS} = I_1 \epsilon^{ij} A_0 \partial_i A_j + I_2 \epsilon^{ij} A_i \partial_0 A_j. \quad (4.17)$$

The coefficient I_2 is shown to be zero, whereas I_1 is given by

$$I_1 = \frac{1}{4} \epsilon^{ij} \text{Tr} \left(S \frac{\partial S^{-1}}{\partial p_i} \left[S \frac{\partial S_0^{-1}}{\partial p_j} \tau^3, S \tau^3 \right] \right). \quad (4.18)$$

Upon performing the traces and the integrations, one finds that $I_1 = 1/(8\pi)$. The coefficient α of Eq. (2.1) is therefore $\frac{1}{2}$, which implies that the statistics of the charged particles of this system is not altered by the presence of the electromagnetic Chern-Simons term.

The superfluid ^3He system exhibits interesting behavior if an external magnetic field $\mathbf{H}$ is applied. In the two-component formalism, the magnetic coupling is $\psi^\dagger \mathbf{H} \cdot \boldsymbol{\sigma} \psi$, and in the four-component formalism, it becomes $\Psi^\dagger \mathbf{H} \cdot \boldsymbol{\sigma} \Psi$. The computation of the effective action in an external magnetic field goes as before, except that the field A_0 is replaced by $A_0 + U \mathbf{H} \cdot \boldsymbol{\sigma} U^{-1}$, and the curl-free condition no longer holds for the total field. Analysis of the planar and A_1 phases can be developed along similar lines.

D. Model with anyons

It is possible to construct a purely academic model with anyons if the asymptotic behavior of $g_3(\mathbf{p})$ matches that of $g_{1,2}(\mathbf{p})$. Consider the model of the previous subsection, but with a nonlocal kinetic term given by $\epsilon(\mathbf{p}) = b|\mathbf{p}| - \mu$. The main difference now is that $g_3(\infty)/|g(\infty)|$ now depends continuously on b and c . The vector $\mathbf{g}$ is $(c\sigma^3 p_1, c\sigma^3 p_2, b|\mathbf{p}| - \mu)$ and

$$\left. \frac{g_3}{|\mathbf{g}|} \right|_{\infty} = \frac{b}{\sqrt{b^2 + c^2}}. \quad (4.19)$$

Therefore the Hopf-term coefficient is

$$\vartheta = \frac{1}{2} \left(\frac{b}{\sqrt{b^2 + c^2}} + 1 \right). \quad (4.20)$$

Let us also give an example of a model in which semions naturally occur. The local, but anisotropic, kinetic term for this model is $\epsilon(\mathbf{p}) = \mathbf{b} \cdot \mathbf{p} - \mu$, where $\mathbf{b}$ is a fixed vector. The restricted formula (3.23) cannot be applied here, since $g_3(\infty)/|g(\infty)|$ depends on the direction of momentum, and the more general formula (3.22) must

be used. The circle of infinite radius on the momentum plane is mapped by the function $\mathbf{g}(p_i)/|\mathbf{g}(p_i)|$ into a curve on $\mathbf{S}_2$ that deviates from the equator, northward if $\mathbf{b} \cdot \mathbf{p} > 0$, and southward if $\mathbf{b} \cdot \mathbf{p} < 0$. The total deviation from the equator is zero, and therefore $\Omega_r = \frac{1}{2}$, like in the relativistic example above. For self-conjugate fermions, this implies that $\vartheta = \frac{1}{2}$, and the solitons are semions.

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APPENDIX A: GAUGE INVARIANCE

In this appendix we provide details of a calculation showing that the only contribution to the induced Hopf term comes from the $B^3 dB^3$ piece of the quadratic part of the effective action. This amounts to verifying the gauge invariance (3.11) of the effective action, before using the curl-free condition.

The relevant pieces of the effective action (3.13) are the two-point and three-point functions,

$$I_{ab}^{\mu\nu\lambda} \int d^3x B_\mu^a \partial_\nu B_\lambda^b + H_{abc}^{\mu\nu\lambda} \int d^3x B_\mu^a B_\nu^b B_\lambda^c, \quad (A1)$$

where the coefficients are

$$I_{ab}^{\mu\nu\lambda} = \frac{1}{2} \text{Tr} \left(S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\mu}, \sigma^a \right\} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\lambda}, \sigma^b \right\} \right), \quad (A2)$$

$$H_{abc}^{\mu\nu\lambda} = \frac{i}{3} \text{Tr} \left(S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\mu}, \sigma^a \right\} S \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\nu}, \sigma^b \right\} \times \frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\lambda}, \sigma^c \right\} \right). \quad (A3)$$

By inspection, one sees that $I_{ab}^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda} \delta_{ab} I_a$ and $H_{abc}^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda} \epsilon^{abc} H$. Let S_0^{-1} be the part of S^{-1} that does not contain σ^3 , and let S_- be the outcome of reversing the sign of σ^3 in S . Then $\frac{1}{2} \left\{ \frac{\partial S^{-1}}{\partial p_\mu}, \sigma^{1,2} \right\} = \frac{\partial S_0^{-1}}{\partial p_\mu} \sigma^{1,2}$ and $S \sigma^{1,2} = \sigma^{1,2} S_-$. We can then write the following expressions for I_a and H :

$$I_3 = \frac{1}{12} \epsilon^{\mu\nu\lambda} \text{Tr} \left(S \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\lambda} \right), \quad (A4)$$

$$I_{1,2} = \frac{1}{12} \epsilon^{\mu\nu\lambda} \text{Tr} \left(S_- \frac{\partial S_0^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S_0^{-1}}{\partial p_\lambda} \right), \quad (A5)$$

$$H = -\frac{1}{18}\epsilon^{\mu\nu\lambda}\text{Tr}\left(S\frac{\partial S_0^{-1}}{\partial p_\mu}S - \frac{\partial S_0^{-1}}{\partial p_\nu}S\frac{\partial S^{-1}}{\partial p_\lambda}\right). \quad (\text{A6})$$

By using the cyclic property of the trace we see that $H = -\frac{2}{3}I_{1,2}$. We can rewrite the effective action as

$$\begin{aligned} \mathcal{L}_{\text{CS}} = & I_3\epsilon^{\mu\nu\lambda}B_\mu^3\partial_\nu B_\lambda^3 + \sum_{a=1,2} I_a\epsilon^{\mu\nu\lambda}B_\mu^a\partial_\nu B_\lambda^a \\ & + 6H\epsilon^{\mu\nu\lambda}B_\mu^1B_\nu^2B_\lambda^3. \end{aligned} \quad (\text{A7})$$

The above form is manifestly gauge invariant since $H = -\frac{2}{3}I_{1,2}$. By using the curl-free condition the only remaining piece is

$$\mathcal{L}_{\text{CS}} = \left(\sum_a I_a + 3H\right)\epsilon^{\mu\nu\lambda}B_\mu^3\partial_\nu B_\lambda^3. \quad (\text{A8})$$

Note that this result does not change if $B_\mu^{1,2}$ do not couple to the fermion with the same (unit) strength as B_μ^3 . Indeed, multiplying $B^{1,2}$ by a constant in the above does not affect the cancellation between H and $I_1 + I_2$.

APPENDIX B: A TOTAL DERIVATIVE

In this appendix we show that the integrand of Eq. (3.17) is a total derivative. Consider the expression

$$I = \epsilon^{\mu\nu\lambda}\text{tr}(S\partial_\mu S^{-1}S\partial_\nu S^{-1}S\partial_\lambda S^{-1}). \quad (\text{B1})$$

In this appendix and the next one, ∂_μ stands for $\partial/\partial p_\mu$. S is an $N \times N$ Hermitian matrix. It is always possible to diagonalize it with an $\text{SU}(N)$ matrix V , i.e., we can write $S^{-1} = V^{-1}DV$ and $S = V^{-1}D^{-1}V$, where D is diagonal. It is then straightforward to show that

$$V(S^{-1}\partial_\mu S)V^{-1} = D^{-1}(iK_\mu D - iDK_\mu + \partial_\mu D), \quad (\text{B2})$$

where we have defined a curl-free, non-Abelian gauge field $K_\mu = -iV\partial_\mu V^{-1}$. If we substitute the above expression in I , many cancellations occur, due to the presence of the antisymmetric tensor, and to the fact that D is diagonal. Only the terms with one or no power of ∂D survive. If we use the curl-free condition on the terms cubic in K and integrate by parts, we find that I is a total derivative:

$$I = -3\epsilon^{\mu\nu\lambda}\partial_\mu\text{tr}(K_\nu DK_\lambda D^{-1} + i\partial_\nu K_\lambda \ln D^2). \quad (\text{B3})$$

APPENDIX C: CALCULATION OF ϑ

In this appendix we show that $(1/|\mathbf{g}|^3)\epsilon^{abc}g_a\partial_1 g_b\partial_2 g_c$ is the Jacobian of the mapping $\mathbf{p} \mapsto \mathbf{g}/|\mathbf{g}|$. For convenience,

we define the unit vector $\hat{\mathbf{g}} = \mathbf{g}/|\mathbf{g}|$. Then

$$\frac{1}{|\mathbf{g}|^3}\epsilon^{abc}g_a\partial_1 g_b\partial_2 g_c = \epsilon^{abc}\hat{\mathbf{g}}^a\partial_1 \hat{\mathbf{g}}^b\partial_2 \hat{\mathbf{g}}^c. \quad (\text{C1})$$

$\hat{\mathbf{g}}$ describes a point on the two-sphere S_2 . Let us introduce coordinates ξ_1, ξ_2 on this sphere. The area element is then given by

$$dS^a = d^2\xi \left(\frac{1}{2}\epsilon_{rs}\epsilon^{abc}\frac{\partial \hat{\mathbf{g}}^b}{\partial \xi_r}\frac{\partial \hat{\mathbf{g}}^c}{\partial \xi_s}\right). \quad (\text{C2})$$

Therefore, we may write

$$\epsilon^{abc}\epsilon^{ij}\hat{\mathbf{g}}^a\frac{\partial \hat{\mathbf{g}}^b}{\partial p_i}\frac{\partial \hat{\mathbf{g}}^c}{\partial p_j}d^2p = \epsilon^{abc}\epsilon^{ij}\hat{\mathbf{g}}^a\frac{\partial \hat{\mathbf{g}}^b}{\partial \xi_r}\frac{\partial \xi_r}{\partial p_i}\frac{\partial \hat{\mathbf{g}}^c}{\partial \xi_s}\frac{\partial \xi_s}{\partial p_j}d^2p. \quad (\text{C3})$$

We therefore explicitly have the Jacobian of the map $\xi_r \mapsto p_i$:

$$\epsilon_{rs}d^2\xi = \epsilon^{ij}\frac{\partial \xi_r}{\partial p_i}\frac{\partial \xi_s}{\partial p_j}d^2p. \quad (\text{C4})$$

Using the above spherical surface element, we can write

$$\vartheta = \frac{1}{4\pi} \sum_r \int \hat{\mathbf{g}}^a dS^a = \frac{1}{4\pi} \sum_r \int dS. \quad (\text{C5})$$

The range of the above spherical integral is found by mapping the boundary of the momentum plane onto S_2 . More specifically, an infinitesimal circle around the origin in momentum space will be mapped onto some closed curve in S_2 , and the circle at ∞ will be mapped onto some other closed curve on S_2 . The coefficient ϑ is the fraction of solid angle comprised between these two curves, summed over diagonal indices.

APPENDIX D: GENERAL ISOSPIN

In this appendix we again consider the simple relativistic model (3.1), but with the generalization that ψ belongs to a general isospin- j representation with generators S^a ($a = 1, 2, 3$). The generators obey the commutation relation $[S^a, S^b] = i\epsilon^{abc}S^c$ and the trace relation $\text{tr}S^a S^b = \kappa\delta^{ab}$, where $\kappa = \frac{1}{3}j(j+1)(2j+1)$. We define the matrix $\varphi = \varphi^a S^a$ and consider the Lagrangian (3.1). After the rotation $\varphi \mapsto S^3$, it becomes

$$\mathcal{L} = \bar{\chi}(i\partial - M - \not{B})\chi, \quad (\text{D1})$$

where $M = m + g_y S^3$, a diagonal matrix. The relevant piece of the quadratic part of the effective action is

$$S_{\text{CS}} = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \text{tr} \left(\frac{\not{p} + M}{p^2 - M^2} \not{B}^3(k) \sigma^3 \frac{\not{p} + \not{k} + M}{(p+k)^2 - M^2} \not{B}^3(-k) \sigma^3 \right). \quad (\text{D2})$$

After a gradient expansion, we find that the coefficient of the Hopf term is

$$\vartheta = 2 \sum_{-j < r < j} (S_{rr}^3)^2 \text{sgn}(M_r) = \frac{2}{3} j'(j' + 1)(2j' + 1), \quad (\text{D3})$$

where j' is the highest value of r such that $m > g_y r$. We may write $\vartheta = 2\kappa'$, where κ is the function of j given above. ϑ is therefore always an integer. For $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, etc., 2κ is respectively equal to 0, 1, 4, 10, 20, 35, etc.

The induced soliton charge comes from the term

$$S_{CS} = ie \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \text{tr} \left(\frac{\not{p} + M}{p^2 - M^2} A(k) \frac{\not{p} + \not{k} + M}{(p+k)^2 - M^2} \not{p}^3 (-k) \sigma^3 \right), \quad (\text{D4})$$

and turns out to be

$$q = \sum_{-j < r < j} S_{rr}^3 \text{sgn}(M_r) = 2(\kappa - \kappa'). \quad (\text{D5})$$

The sum $\vartheta + q$, which is the net statistics, is simply the coupling independent constant 2κ .

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