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The Thirring and the Veneziano models*

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Motivated by the development of a relativistic string, we extend the Thirring model in two-dimensional space-time and solve for its Wightman functions. By conjecturing that the scale dimensions of the Thirring fields are the momenta squared in physical space, we are able to introduce a dimensional scale, the Regge slope α' . We interpret the Thirring fields to be constituent fields such that a physical hadron field is a product of appropriate Thirring fields. Then it follows that the vacuum expectation values (i.e., the integrand) in the Veneziano model become Wightman functions in our extended Thirring model. Throughout, emphasis is placed on the conviction that, for physically observable quantities, currents are more fundamental than constituent fields.

I. INTRODUCTION

One of the most distinct features of strong interactions is the existence of numerous hadrons and the absence (to this moment) of any fundamental constituents from which all hadrons are composed. This "nuclear democracy" feature has hindered the traditional field-theoretical approach from being a meaningful tool to study hadron physics. Hence the S -matrix approach has received considerable attention in the past years. It is in this spirit that Veneziano¹ first proposed a simple beta function as a four-point scattering amplitude. Since then the model has been generalized to N -

point functions and the operator formalism has been introduced to study the hadron spectrum.² It is found that the spectrum degenerates exponentially as a function of the mass.³

It was noted by Nambu and others⁴ that the spectrum of the dual resonance model (DRM) is identical to a vibrating string. Later Nambu⁵ proposed an action in terms of the area swept out by the string. This formulation is studied by Chang and Mansouri⁶ and more recently, by Goddard, Goldstone, Rebbi, and Thorn (GGRT).⁷

One of the most significant roles of the string is that it provides a bridge between the field-theoretical and the S -matrix approaches to strong inter-

actions. In fact this has led many⁸ to investigate two-dimensional field-theoretic models with conformal symmetry. It is discovered that the conformal transformation generators give the Virasoro algebra⁹ in dual models. Eigenstates of the dilatation operator turn out to be the physical hadron states. This opens the program of investigating other conformally invariant two-dimensional Lagrangians in connection with the dual models. In particular, Kikkawa¹⁰ and Andric and Gervais¹⁰ have attempted to relate the Thirring model to the dual models. Both works accept the original DRM framework. Kikkawa uses the Thirring model to generalize the dual model in the direction suggested earlier by Neveu and Schwarz,¹¹ while Andric and Gervais use the Thirring model to add a fifth space-time-like dimension to the DRM so that the Regge intercept can be shifted.¹²

The main purpose of our work is also to relate the Thirring model to the DRM, and to the Virasoro-Shapiro model¹³ as well. However, here we obtain the complete DRM from an extended Thirring model. Our motivation is best summarized by Fubini, Hanson, and Jackiw (FHJ),¹⁴ who conjecture that a new theoretical scheme for hadron physics may be based on the following two postulates: First, the field theory should be confined to two space-time dimensions; second, the dimensionality in the field theory should be proportional to the mass squared M^2 .¹⁵ For definiteness, the two-dimensional space is called the parameter space (or, maybe more appropriately, the auxiliary space), in distinction to the physical Minkowski space.

In this paper we provide an explicit realization of the above postulates, which, for the purpose of reference, will be called the duality postulates. Our intentions are primarily threefold: first, to show that the two-dimensional Lagrangian with four spinor fields ($\mu = 0, 1, 2, 3$, with Minkowski metric) in the parameter space, $\mathcal{L} = \bar{\psi}^\mu \not{\partial} \psi_\mu$, has a spectrum identical to the vibrating string if we treat the currents as coordinates obeying canonical commutation relations and if the energy-momentum tensor vanishes (we recall that the energy-momentum tensor of a free string vanishes, either as a consequence of the invariance of reparametrization or as an added constraint depending on its formulation); second, to introduce the extended Thirring interactions to the above Lagrangian, and to obtain a more precise understanding of the duality postulates by presenting explicitly the connection between the extended Thirring model and the DRM; and third, to make an attempt to probe into the underlying constituent structure of the hadrons in the DRM.

The spinor field representation of a free string is introduced because it provides a better framework to introduce interactions. Implicit in the duality postulates is that the field theory should have conformal symmetry. The (naive) dimension of the spinor fields is $\frac{1}{2}$, so that the only conformally invariant interaction involving only fermions is the Thirring interaction. Hence the choice is rather unique. In the usual string formalism, the fields have zero dimension. Thus we have the ambiguity that any conformally invariant interaction remains so after being multiplied by a function of the fields. The spinor formalism also will explain, at least in part, why the vertex operator in the DRM is a natural interaction of the string.

The explicit connection between the extended Thirring model and the dual models allows the refinement of the duality postulates:

(1) The field theory is confined to two space-time dimensions with conformal symmetry. The vacuum and the multiparticle states in the parameter space become the vacuum and the single-hadron states in the physical space, respectively.

(2) The dimensionality in the parameter space is proportional to the mass squared M^2 in the physical space, where the proportionality constant is a dimensional scale, namely, the Regge slope α' .

We may go even further by postulating the following:

(3) Appropriate Green's functions in the parameter space can be interpreted as Green's functions in the physical space.

In many approaches to particle physics, such as quark-parton models, the underlying structure has to be determined first before one can construct scattering amplitudes. However, in the dual models, very rich spectra and algebraic properties are obtained from relatively simple scattering amplitudes. In our probe into the structure of the hadron fields in the DRM, the Thirring fields become mathematical constituent fields that build up the hadron fields and the scattering comes from the exchange of constituents. We show that the scattering amplitudes are relatively independent of the detailed structures of the hadron fields. As an example, the same hadron field can be built up by one pair or two pairs of constituent fields.

The reader should be warned that, throughout this paper, we have assumed that Wick's rotation is valid in the parameter space in which the proper-time quantization¹³ is carried out. The physical space however is *always* kept Minkowskian.

The organization of this paper is as follows. In Sec. II we review the properties of the DRM and the Virasoro-Shapiro model that are relevant for later discussions. In Sec. III we review the free vibrating string with emphasis on its mathematical aspects. In so doing, we lose the clear classical picture and hence some of the steps taken become rather obscure. However, with the prior knowledge of the physical picture of Nambu and GGRT,⁷ we feel certain of the procedure since we arrive at identical results.

In Sec. IV the string problem is recast in the language of currents only. The idea of treating currents as coordinates obeying canonical commutation relations is an old one.^{16,17} The resulting constraints and equation of motion can be solved, giving us back trivially the string spectrum and all of its properties. At this point, the question concerning which is more fundamental, the currents or the fields, is purely academic. But when we investigate the free spinor field Lagrangian in the parameter space, we regain the string spectrum only when the currents are treated as fundamental. Next we turn on the Thirring's spinor interaction and observe that the spectrum remains unchanged.

In Sec. V we review the solutions to the Thirring model following closely the work of Dell'Antonio, Frishman, and Zwanziger (DFZ).¹⁸ The necessary extensions of the Thirring model are straightforward. We introduce extra degrees of freedom such that we have four spinor fields instead of one. However, because of the indefinite metric accompanying this extra degree of freedom, ghost states will in general be present. To eliminate them, we have to impose constraints on the fields. It turns out that the vanishing of the energy-momentum tensor is sufficient.¹⁹ Another extension is the introduction of spinor fields with continuous spin. The fact that this is a zero-mass two-dimensional theory allows us to do precisely this.

In Sec. VI the integrands of the Veneziano amplitudes are obtained from Thirring Green's functions via the duality postulates. The Harari-Rosner dual quark diagrams²⁰ suggest that the hadron fields can be expressed as a product of two appropriate Thirring fields. Therefore, the Thirring Green's function of eight Thirring fields becomes the Green's function of four hadron fields. It is observed that the light-cone singularities from products of operator fields are closely associated with the poles in the DRM.

In Sec. VII we examine hadron fields with different products of Thirring fields and come to the conclusion that the hadron field is quite independent of its detailed underlying structure. Section VIII contains some remarks.

Readers who are not particularly interested in the string may go directly from Sec. II to Sec. V.

II. REVIEW OF THE DUAL MODELS

Here we review the basic properties of the DRM² that are relevant to subsequent discussions. We consider only the case where the Regge intercept $\alpha_0 = 1$. The scattering amplitude of N external ground-state particles (of momentum k_i) can be written as

$$B_N(k_1, \dots, k_N) = \int \frac{dz_1 \cdots dz_N}{d\Omega} \prod_{i,j} |z_i - z_j|^{2k_i \cdot k_j}, \quad (2.1)$$

where the invariant differential volume

$$d\Omega = - \frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \quad (2.2)$$

allows any three z_i 's to be set to constants. The amplitude can also be written in an operator formalism:

$$B_N = \int \frac{dz_1 \cdots dz_N}{d\Omega} \langle 0 | R \left[\prod_{j=1}^N \frac{V(z_j, k_j)}{z_j} \right] | 0 \rangle, \quad (2.3)$$

where R symbolizes the $|z|$ -ordered product and

$$V(z, k) = \exp[i\sqrt{2}k \cdot Q^{(+)}(z)] \exp[i\sqrt{2}k \cdot Q^{(0)}(z)] \times \exp[i\sqrt{2}k \cdot Q^{(-)}(z)], \quad (2.4a)$$

$$Q(z) = Q^{(+)}(z) + Q^{(0)}(z) + Q^{(-)}(z), \quad (2.4b)$$

$$Q_\mu^{(+)}(z) = \sum_{n=1}^{\infty} \frac{a_{n\mu}}{n} z^n, \quad (2.4c)$$

$$Q_\mu^{(-)}(z) = \sum_{n=1}^{\infty} \frac{a_{n\mu}}{n} z^{-n}, \quad (2.4d)$$

$$Q_\mu^{(0)}(z) = q_\mu - ip_\mu \ln z, \quad (2.4e)$$

and

$$[a_{n\mu}, a_{m\nu}^\dagger] = \delta_{n,m} g_{\mu\nu}, \quad (2.4f)$$

$$[q_\mu, p_\nu] = i g_{\mu\nu}. \quad (2.4g)$$

The metric $g_{\mu\nu}$ is defined as $g_{11} = g_{22} = g_{33} = -g_{00} = 1$. Introducing the expression

$$P_\mu(z) = z \frac{d}{dz} Q_\mu(z), \quad (2.5a)$$

we see that the following commutation relations can be verified to be true for $|z| = |z'|$, where $z = re^{i\theta}$:

$$[Q_\mu(z), P_\nu(z')] = 2\pi g_{\mu\nu} \delta(\theta - \theta'), \quad (2.5b)$$

$$[P_\mu(z), P_\nu(z')] = 2\pi i g_{\mu\nu} \delta'(\theta - \theta'). \quad (2.5c)$$

The vacuum is defined by $a_n|0\rangle = p|0\rangle = 0$. Let us

now introduce the operators L_n :

$$L_n = \frac{i}{4\pi} \oint dz z^{n-1} : P^2(z) : , \quad (2.6)$$

where the dots refer to normal ordering and the integral is evaluated along a small circle of the complex plane z with the center at the origin. It is straightforward to show that, for integers n and m , the following algebra for L_n is true:

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0} \frac{n(n^2-1)}{3}. \quad (2.7)$$

This is called the Virasoro (or conformal) algebra.⁹ The transformation properties of the vertex operator $V(z, k)$ under L_n are given by the following commutation relation:

$$\left[L_n, \frac{V(z, k)}{z} \right] = z^n \frac{d}{dz} V(z, k) + n\alpha' k^2 z^{n-1} V(z, k), \quad (2.8)$$

where the right-hand side becomes a total derivative since $\alpha' k^2 = 1$. Using Eq. (2.8), $V(z, k) = z^{L_0} V(1, k) z^{-L_0}$, and the scattering amplitude for a given ordering in z , $|z_i| \geq |z_{i+1}|$, can be written as

$$\left\langle k_1 \left| V(1, k_2) \frac{1}{L_0 - 1} V(1, k_3) \cdots \frac{1}{L_0 - 1} V(1, k_{N-1}) \right| k_N \right\rangle, \quad (2.9)$$

where

$$\begin{aligned} |k\rangle &= \lim_{z \rightarrow 0} \frac{V(z, k)}{z} |0\rangle \\ &= \oint \frac{V(z, k)}{z^2} dz |0\rangle \\ &= e^{i\sqrt{2}k \cdot q} |0\rangle, \end{aligned} \quad (2.10a)$$

$$\langle k| = \lim_{z \rightarrow \infty} \langle 0| V(z, k) z = \langle 0| e^{i\sqrt{2}k \cdot q}. \quad (2.10b)$$

The vertex operator $V(z, k)$ has the following properties:

$$V^\dagger(z, k) = V\left(\frac{1}{z}, -k\right) \quad (2.11)$$

and

$$\begin{aligned} V(z, k) V(z', k') &= V(z', k') V(z, k) \\ &\times \exp[2\pi i k \cdot k' \eta(z - z')], \end{aligned} \quad (2.12)$$

where

$$\eta(z - z') = \frac{z - z'}{|z - z'|}.$$

Note that the set $L_{\pm 1}$ and L_0 is also closed. They form the projective group. The amplitude B_N is invariant under the projective transformation and

hence is invariant under cyclic permutation. This can be shown using properties (2.8) and (2.12). Factorization is manifest in Eq. (2.9). The complete set of states is

$$|\lambda, k\rangle = \left[\prod_j \frac{(a_{j\mu}^\dagger)^{\lambda_{j\mu}}}{(\lambda_{j\mu}!)^{1/2}} \right] e^{i\sqrt{2}k \cdot q} |0\rangle. \quad (2.13)$$

Because of the indefinite space-time metric $g_{\mu\nu}$, some of the states have negative norms. These states are called ghost states. Their presence is inevitable because we have demanded explicit Lorentz covariance.

Luckily it has been shown that no ghost states can satisfy the following conditions¹⁹:

$$\langle \lambda', k' | L_n | \lambda, k \rangle = 0, \quad n \neq 0 \quad (2.14)$$

$$L_0 | \lambda, k \rangle = +\alpha_0 | \lambda, k \rangle. \quad (2.15)$$

This fact, together with Eq. (2.8), enables us to conclude that all ghost states are effectively decoupled (i.e., spurious). Equation (2.14) gives Gupta-Bleuler-type subsidiary constraints and Eq. (2.15) is an on-mass-shell statement.

Equation (2.10) provides us the projection of the ground state from the vertex. In fact, the operators associated with the states on the leading trajectory have also been derived.^{21,22}

Removing the $d\Omega$ from Eq. (2.1), we have for B_N the following expression:

$$B_N = \int \prod_{j \neq a, b, c} dz_j \prod_{\substack{i > j \\ i \neq c}} |z_i - z_j|^{2k_i \cdot k_j} \quad (2.16)$$

Shapiro¹³ has proposed a model for the N -point function with the Regge intercept $\alpha_0 = 2$:

$$G_N = \int \prod_{j \neq a, b, c} d^2 z_j \prod_{\substack{i > j \\ i \neq c}} |z_i - z_j|^{2k_i \cdot k_j}. \quad (2.17)$$

The only difference between Eq. (2.16) and Eq. (2.17) is the domain of integration of z_j . For G_N , the integration volume is the entire complex z_j plane. When $N=4$, it reduces to the Virasoro model.¹³ The operator formalism for G_N has been worked out,²³ using essentially the same techniques in the DRM. In short, a completely new set of oscillators is introduced. If $\bar{z}$ is the conjugate of z , we have

$$Q_\mu(\bar{z}) = \bar{q}_\mu - i\bar{p}_\mu \ln \bar{z} + \sum_{n=1}^{\infty} \frac{b_{n\mu}}{n} \bar{z}^{-n} + \sum_{n=1}^{\infty} \frac{b_{n\mu}^\dagger}{n} \bar{z}^n, \quad (2.18)$$

where $Q_\mu(\bar{z})$ commutes entirely with $Q_\mu(z)$ and has identical properties to $Q_\mu(z)$. It follows immediately that there is a new Virasoro algebra $\bar{L}_n$ identical to Eq. (2.7). The vertex operator becomes

$$V(z, \bar{z}, k) = V(z, k)V(\bar{z}, k). \quad (2.19)$$

Since the physical states must have the same eigenvalue for the momentum operators $\bar{p}$ and $\bar{p}$, the presence of this doubling is only apparent. In fact, the DRM propagator $(L_0 - 1)^{-1}$ changes to

$$\frac{2}{L_0 + \bar{L}_0 - 2} \frac{\sin \pi(L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} \quad (2.20)$$

in the Virasoro-Shapiro model. Here the ordering is in $r = (z\bar{z})^{1/2}$, which plays the analogous role of time in the conventional Lehmann-Symanzik-Zimmermann (LSZ) formalism.

We end with the remark that the degeneracy of the spectra of both models is of the Hagedorn type²⁴; that is, the number of states d_N at mass squared $M^2 = N$ for large N is

$$d_N \sim \exp(\text{constant} \times \sqrt{N}). \quad (2.21)$$

III. REVIEW OF THE STRING

The interest in a relativistic string²⁵ was first brought up by Nambu⁴ when he observed that the DRM spectrum bears a striking resemblance to that of a vibrating string in Minkowski space where the amplitudes of vibration are taken as space-time coordinates. Nambu⁵ further suggested that the action of the string should be proportional to the area of the "world sheet" swept out by it in space-time. To describe the surface swept out, we need only two intrinsic variables. We call them σ and τ so that the space-time coordinate X_μ is a function of them: $X_\mu = X_\mu(\sigma, \tau)$. It is clear that a change of parameterization will not change the surface. The Lagrangian is of the form

$$\mathcal{L} = \left[- \left(\frac{\partial X}{\partial \sigma} \right)^2 \left(\frac{\partial X}{\partial \tau} \right)^2 + \left(\frac{\partial X}{\partial \tau} \frac{\partial X}{\partial \sigma} \right)^2 \right]^{1/2}. \quad (3.1)$$

To quantize the string, there are in general two approaches, namely covariant and noncovariant quantizations.

Recently, GGRT⁷ undertook the second approach, namely the noncovariant quantization, in a very elegant and lucid paper. The physical picture of the string emerges clearly and its space-time interpretation is well defined. Since they never quantize the timelike component, no "ghost" is present; however, Lorentz covariance needs to be verified and GGRT show that this is possible only for intercept $\alpha_0 = 1$ and in 26 space-time dimensions. The beauty of their approach of quantization is that each step in the development is well motivated by the physical picture. Here we would rather review the covariant quantization procedure, where Lorentz covariance is explicit but, because of the indefinite metric of Minkowski space, it is the no-ghost property that needs to

be demonstrated. This formalism has been studied by many authors.^{6,8} Here the physical picture is not as clear as that in the GGRT work. However, it permits the quantization of the string in four space-time dimensions. Throughout this section, we emphasize the fact that the Lagrangian is a well-defined two-dimensional Lagrangian in the parameter space. We hope this can bring out more transparently the mathematical content of the quantization procedure, which is relevant to subsequent discussions.

The parameter space has coordinates σ and τ , while the physical space has coordinates X_μ . In the parameter space, X_μ is a scalar field with dimension zero and so the two-dimensional Lagrangian (3.1) is conformally invariant. The metric in the parameter space is g_{ab} [$a=0, 1$ (for τ and σ)]. The physical space-time index μ describes some unspecified additional degree of freedom which is to be summed over with metric $g_{\mu\nu}$ when repeated. Only when we go to the physical space do we give the usual space-time interpretation to this index. Note that the terms covariant and noncovariant quantization refer to the physical space.

The arbitrariness of parameterization of the free-string Lagrangian (3.1) allows us to choose an orthonormal parameterization such that (sum over μ implied)⁶⁻⁹

$$\frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \tau} = 0 \quad (3.2a)$$

and

$$\left(\frac{\partial X}{\partial \tau} \right)^2 + \left(\frac{\partial X}{\partial \sigma} \right)^2 = 0. \quad (3.2b)$$

The Lagrangian (3.1) then becomes

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial X}{\partial \tau} \right)^2 - \frac{1}{2} \left(\frac{\partial X}{\partial \sigma} \right)^2, \quad (3.3)$$

from which follows the equation of motion

$$\square X_\mu = \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X_\mu = 0. \quad (3.4)$$

Introducing the light-cone variables⁶

$$u = \sigma + \tau,$$

$$v = \sigma - \tau,$$

Eqs. (3.2) and (3.4) become

$$\frac{\partial X}{\partial u} \frac{\partial X}{\partial u} = \frac{\partial X}{\partial v} \frac{\partial X}{\partial v} = 0, \quad (3.5)$$

$$\partial_u \partial_v X_\mu = 0. \quad (3.6)$$

Solving Eq. (3.6) gives⁶

$$X = X_+(u) + X_-(v) \quad (3.7)$$

and Eq. (3.5) becomes

$$\frac{\partial X_+}{\partial u} \frac{\partial X_+}{\partial u} = \frac{\partial X_-}{\partial v} \frac{\partial X_-}{\partial v} = 0. \quad (3.8)$$

This allows us to consider X_+ and X_- separately. For reasons that will become clear later on, we want to bring out the conformal symmetry property. To do so we follow the Lorentz-invariant quantization method of FHJ,¹⁴ where the dilatation operator D is the evolution operator in proper time r ,

$$r^2 = \sigma^2 - \tau^2.$$

Because of scale invariance, D is independent of r . Following FHJ,¹⁴ we assume the validity of Wick's rotation of τ and discuss the quantization in the resulting Euclidean space ($\tau \rightarrow i\tau$). The equal-proper-time canonical commutation relation to be imposed is

$$\left[\frac{d}{d(\ln r)} X_{\pm}(r, \theta), X_{\pm}(r, \theta') \right] = 2\pi \delta(\theta - \theta'). \quad (3.9)$$

Since

$$r \frac{d}{dr} = \frac{d}{d(\ln r)} = u \frac{d}{du} + v \frac{d}{dv},$$

where

$$u = r e^{i\theta},$$

$$v = r e^{-i\theta},$$

Eq. (3.9) for X_+ becomes

$$\left[u \frac{d}{du} X_+(u), X_+(u') \right]_{r=r'} = 2\pi \delta(\theta - \theta'). \quad (3.10)$$

The factor 2π is introduced to conform to previous conventions in DRM. We can also differentiate Eq. (3.10) with respect to r' to obtain $[\dot{X}_+(u) = u(d/du)X_+(u)]$

$$[\dot{X}_{+\mu}(u), \dot{X}_{+v}(u')]_{r=r'} = 2\pi i g_{\mu\nu} \delta'(\theta - \theta'), \quad (3.11)$$

where the extra degree of freedom μ is shown explicitly. The right-hand side of Eq. (3.10) is real. This means that $X_+(r, \theta)$ cannot be a Hermitian operator. FHJ postulate that the field $X_+(r, \theta)$ should satisfy²⁶

$$X_+^\dagger(r, \theta) = X_+\left(\frac{1}{r}, \theta\right). \quad (3.12)$$

$X_-(v)$ can be treated identically.

There is one point that needs clarification. In this treatment we have not imposed any boundary conditions on the string (in the physical space). Therefore two independent solutions are available, namely X_+ and X_- . However, if the string has free ends, the constraints at the ends allow only one independent solution X_r , which is a linear

combination of $X_{\pm}$, and it obeys

$$[X_r(\sigma, \tau), X_r(\sigma', \tau)] = 0, \quad (3.13a)$$

$$\left[\frac{\partial}{\partial \tau} X_r(\sigma, \tau), X_r(\sigma', \tau) \right] = \delta(\sigma - \sigma'). \quad (3.13b)$$

Since we are primarily interested in the interactions of the string, we do not impose the free-ends constraint on the string. In fact, a free closed string has both sets of modes. Equation (3.8) allows us to consider $X_{\pm}$ separately, so the quantization of both $X_{\pm}$ is no more complicated than that of a string with free ends.

Comparing Eqs. (3.10) and (3.11) with Eq. (2.5), we can immediately write down the solution of $X_{+\mu}$,

$$X_{+\mu}(z) = Q_{\mu}(z). \quad (3.14)$$

This satisfies Eqs. (3.6) and (3.12), which can be shown easily, using Eq. (2.4). In the parameter space, the many-particle states are given by

$$|\lambda\rangle = Z(\lambda) \prod_i \frac{(a_{i\mu}^\dagger)^{\lambda_{i\mu}}}{(\lambda_{i\mu}!)^{1/2}} |0\rangle, \quad (3.15)$$

where $Z(\lambda)$ is some phase factor and the vacuum is defined by

$$p_{\mu} |0\rangle = a_{n\mu} |0\rangle = 0. \quad (3.16)$$

However, in the physical space, these many-particle states become excited modes of the quantized string. In the DRM, they become resonance states [Eq. (2.13)]. The vacuum state in the parameter space becomes the vacuum state in the physical space. In this covariant quantization procedure the physical motivation for this correspondence is obscure. We regard this as a postulate. However, in the noncovariant quantization it is physically clear why this happens.

Now we still have to take into account the constraint Eq. (3.8). This becomes an operator statement and has to be imposed weakly:

$$\left\langle \lambda_1 \left| \frac{\partial X_+}{\partial u} \frac{\partial X_+}{\partial u} \right| \lambda_2 \right\rangle = 0. \quad (3.17)$$

Normal ordering is necessary for Eq. (3.8), and this will in general introduce an extra c number to the right-hand side of Eq. (3.17).

Identifying

$$X_+(u) = Q(u),$$

$$\dot{X}_+(u) = P(u),$$

we obtain the Virasoro gauge conditions (2.14) and (2.15). It has been proved by Brower, and Goddard and Thorn¹⁹ that there is no ghost state in the physical Fock space. By treating the other solution $X_-(v)$ identically, the covariant quantization of the string is complete. However, the non-covariant quantization is possible only in 26 di-

mensional space (with 25 space coordinates), while here the quantization is possible in physical Minkowski space, or any space with less than or equal to 26 dimensions. Also, in the noncovariant quantization, $\alpha_0 = 1$ emerges when we impose Lorentz covariance on the free string. Here, this condition emerges only when the interaction vertex operator is introduced. In fact, we believe it is impossible to quantize the free string covariantly without any knowledge of the vertex operator. So, for some unknown reason, the interaction vertex in the DRM is the natural interaction of the string. We will come back to discuss this point in the last section.

IV. SPINOR REPRESENTATION OF A RELATIVISTIC STRING

The search for physical quarks or other constituents which build up hadrons has been unsuccessful so far. Among the numerous known hadrons none plays the fundamental role of a "building block" or an elementary particle. This has led many to accept the idea of "nuclear democracy" or "bootstrap." Hence, to study strong interactions with relativistic field theory, one is hindered by the difficulty of choosing canonical fields. The remarkable success of current algebra²⁷ in correlating various properties of strong interactions has led Dashen and Sharp¹⁶ to postulate that a complete theory of hadrons may be described only in terms of currents and their commutation relations. One of their key observations is that currents treat all particles *equally*. To check the consistency of this idea with the field-theoretical formalism of the string, we first express the string Lagrangian in terms of currents. We then show the same can be done with the free spinor fields in the parameter space, giving back the free-string spectrum. Because of the dimension $\frac{1}{2}$ of the spinor field in a two-dimensional space, the interaction term which is conformally invariant is rather unique, namely, the Thirring model. Turning on this interaction does not change the free-string spectrum.

The energy-momentum tensor of the Lagrangian (3.1) vanishes identically¹⁴:

$$\begin{aligned} \Theta_s^{ab} = & \frac{1}{\mathcal{L}} [\partial_c X^\mu \partial^c X^\nu (\partial^a X_\mu \partial^b X_\nu - \frac{1}{2} g^{ab} \partial_f X_\mu \partial^f X_\nu) \\ & - \partial_c X^\mu \partial^c X_\mu (\partial^a X_\nu \partial^b X^\nu - \frac{1}{2} g^{ab} \partial_f X^\nu \partial^f X_\nu)] \\ = & 0. \end{aligned} \quad (4.1)$$

(We recall that the Roman indices refer to the coordinates in the parameter space and the Greek indices are the physical space-time indices.) On the other hand, the "free-boson-field"-like La-

grangian Eq. (3.3) also has a vanishing energy-momentum tensor Θ_B^{ab} , this time due to the constraints (3.2):

$$\begin{aligned} \Theta_B^{ab} = & \partial^a X^\mu \partial^b X_\mu - \frac{1}{2} g^{ab} \partial^c X^\mu \partial_c X_\mu \\ = & 0. \end{aligned} \quad (4.2)$$

This is traceless and symmetric and therefore has only two independent components,

$$\begin{aligned} \Theta_B^{00} = & \left(\frac{\partial X}{\partial \tau} \right)^2 - \frac{1}{2} \left[\left(\frac{\partial X}{\partial \tau} \right)^2 - \left(\frac{\partial X}{\partial \sigma} \right)^2 \right] \\ = & \frac{1}{2} \left(\frac{\partial X}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial X}{\partial \sigma} \right)^2 \\ = & 0, \\ \Theta_B^{01} = & \frac{\partial X}{\partial \tau} \frac{\partial X}{\partial \sigma} = 0. \end{aligned} \quad (4.3)$$

The four scalar fields X_μ are not independent and the constraint relating them is simply the vanishing of the energy-momentum tensor.

Equation (3.4) allows us to define a current

$$j_a(\sigma, \tau) = \partial_a X(\sigma, \tau) \quad (4.4)$$

which is conserved,

$$\partial^a j_a(\sigma, \tau) = \square X(\sigma, \tau) = 0. \quad (4.5)$$

In fact

$$\begin{aligned} j_+(u, v) = & j_0(u, v) + j_1(u, v) \\ = & j_+(u) \\ = & \partial_u X_+(u), \\ j_-(u, v) = & j_0(u, v) - j_1(u, v) \\ = & j_-(v) \\ = & \partial_v X_-(v). \end{aligned} \quad (4.6)$$

The commutation rules are defined as (with μ index suppressed)

$$[j_+(u), j_+(u')] = 2ic \delta'(u - u'), \quad (4.7a)$$

$$[j_-(v), j_-(v')] = 2ic \delta'(v - v'), \quad (4.7b)$$

$$[j_+(u), j_-(v)] = 0, \quad (4.7c)$$

where the normalization constant c is determined by Eq. (3.10). The energy-momentum tensor becomes

$$\Theta_B^{ab} = j^a j^b - \frac{1}{2} g^{ab} j^c j_c. \quad (4.8)$$

The vanishing of Θ_B^{ab} reproduces the constraints (3.8):

$$j_+^2(u) = j_-^2(v) = 0. \quad (4.9)$$

Now we can go ahead to solve and quantize Eqs. (4.5) and (4.9), which obviously reproduce the string spectrum. We note that j_a satisfies the

zero-mass Klein-Gordon equation and hence can be decomposed into positive and negative frequencies in an invariant manner so that the normal ordering of the product of currents in Eq. (4.9) is well defined. Of course, such normal ordering will in general introduce a c number.

With these trivial preliminaries, we can now look at the two-dimensional Lagrangian with four free spinor fields

$$\mathcal{L} = \frac{1}{2} \bar{\psi}^\mu \gamma^a \partial_a \psi_\mu. \quad (4.10)$$

For the rest of this paper, we choose the following set of matrices, with γ_5 diagonal:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.11)$$

The equation of motion is

$$\gamma \cdot \partial \psi(\sigma, \tau) = 0. \quad (4.12)$$

We can define a vector current $j_a(\sigma, \tau) = \bar{\psi} \gamma_a \psi$ and an axial-vector current $j_{5a} = \bar{\psi} \gamma_a \gamma_5 \psi$, both of which are conserved, as can be shown from the equation of motion (4.12):

$$\partial^a j_a = \partial^a j_{5a} = 0. \quad (4.13)$$

In fact, from Eq. (4.12), we obtain for

$$\psi(u, v) = \begin{pmatrix} \psi_1(u, v) \\ \psi_2(u, v) \end{pmatrix}$$

the following:

$$\begin{aligned} \psi_1 &= \psi_1(u), \quad \psi_2 = \psi_2(v), \\ j_+^\mu(u, v) &= 2\bar{\psi}_1^\mu(u) \psi_1^\mu(u) \quad (\mu \text{ not summed}) \\ &= j_+^\mu(u), \\ j_-^\mu(u, v) &= j_-^\mu(v) = 2\bar{\psi}_2^\mu(v) \psi_2^\mu(v). \end{aligned} \quad (4.14)$$

Instead of defining the commutation rules for the spinor fields, we define the commutation rules for the currents j , which are given by Eq. (4.7). The energy-momentum tensor is identical to Eq. (4.8),²⁸ namely,

$$\Theta_F^{ab} = j^a j^b - \frac{1}{2} g^{ab} j^c j_c. \quad (4.15)$$

With the interpretation that follows Eq. (4.3), we set Θ_F^{ab} equal to zero, obtaining the constraint (4.9). With current conservation, we can introduce a set of zero-mass free scalar fields such that

$$j_{a\mu} = \partial_a \varphi_\mu(\sigma, \tau). \quad (4.16)$$

We collect all the equations:

$$\begin{aligned} \Theta_F^{ab} &= 0, \\ \square \varphi_\mu &= 0, \\ \square j_{a\mu} &= 0. \end{aligned}$$

Together with the current commutation rules (4.7), we see that the quantized solution is again that of the string.

Actually, in this free-spinor case, one can introduce equal- τ (proper-time) canonical anticommutation rules for the spinor fields. The current commutation rules Eq. (4.7) then follow from the definition of currents in terms of the fields. The conventional definition of the energy-momentum tensor gives

$$\Theta_c^{ab} = \frac{1}{4} (\bar{\psi} \gamma^a \bar{\partial}^b \psi + \bar{\psi} \gamma^b \bar{\partial}^a \psi). \quad (4.17)$$

The quantization gives the spinor field linear in the anticommuting oscillator b_{ii} . Therefore Θ_c^{ab} is bilinear in b_{ii} , while Θ_F^{ab} is quadrilinear in b_{ii} . However, a straightforward calculation shows that Θ_F^{ab} reduces to Θ_c^{ab} in the free-spinor case. (In fact, the possibility of constructing zero-mass Bose fields from bilinear combinations of zero-mass fermion fields was known as early as 1935.²⁹) However, there are two essential differences between the forms (4.15) and (4.17).

As we shall see, when a conformally invariant interaction term is introduced, Θ_c^{ab} changes but Θ_F^{ab} retains its present form. The dimension of the fields changes, while that of the currents remains canonical. Second, the external coordinate q and momentum p emerge naturally from the quantization of j and its integrated forms when the current commutation rules are imposed. What we mean here is that the zero moment (in z) of $zj(z)$ obtained from its spinor representation cannot be interpreted as a momentum operator because the corresponding coordinate q cannot be constructed. This means we have to introduce by hand another zero-moment term for $zj(z)$ if we use Θ_c^{ab} .

The dilatation operator is the evolution operator in the parameter space and the many-particle states are its eigenstates with corresponding eigenvalues. Since these eigenstates are also the physical resonance states which should be independent of the parameter-space proper time τ , we want to preserve scale invariance as a symmetry when we introduce an interaction term to the Lagrangian so that the dilatation operator is independent of τ .

We can, of course, introduce a conformally invariant interaction term to the original free-string Lagrangian. The dimension of the scalar field is zero, so that a likely candidate probably has the following form:

$$\mathcal{L}_{\text{interaction}} = G(X) \partial_b X \partial^b X$$

for any function $G(X)$. At this moment, we do not know which interaction to look into. In fact, when X has only one component, this gives back the massless Klein-Gordon equation with no inter-

action. In general, the interaction is such that one can only solve it perturbatively. On the other hand, the Thirring model can be solved exactly and has been studied in great detail.³⁰ This is one reason why we go to all the trouble to introduce the two-dimensional spinor field.

V. THE EXTENDED THIRRING MODEL

It was Thirring³⁰ who first realized that the self-interaction of a zero-mass spinor field in two-dimensional space leads to an explicitly soluble model. However, the early history of the model was a rather confusing one³⁰ until Johnson³¹ started from scratch by abandoning the canonical commutation relation for the spinor fields and emphasizing the role of the conserved currents. He gives the commutation rules between the currents and the fields. His currents are defined in terms of products of fields. This definition is complicated by the need to separate the points in the fields by an infinitesimal amount. Later Callan, Dashen, and Sharp²⁸ suggested that one should be able to describe the model in terms of the conserved currents and their commutation relations. More recently, Dell'Antonio, Frishman, and Zwanziger (DFZ),¹⁸ in an elegant paper, carried out this idea further by writing down Wightman functions for the spinor fields as vacuum expectation values of operators depending only on the currents. Their derivation shows that the Thirring model is determined by the current commutators and the commutators between currents and fields. The equation of motion and the commutation relations of the spinor fields are consequences of the irreducibility assumption about the currents.

In this section the Thirring model is extended in two directions and its solution is derived. The presentation is complete enough that we hope the reader can supply the derivations. We follow very closely the work of DFZ,¹⁸ where most of the details can be found. Some of the formulas in the previous sections are repeated here to expedite the reading. x is the space-time coordinate in the parameter space. Throughout this section, repeated greek indices are *not* summed unless explicitly stated. We also assume the validity of Wick's rotation in the parameter space. Many of the integrals become well defined only in the Euclidean parameter space. Also, the c -number Schwinger term in the commutator of the energy-momentum tensors is finite only in the Euclidean space, where the contour of the integrals (from negative infinity to positive infinity) can be deformed to a circle around the origin. See FHJ¹⁴ for a discussion on this point.

As we have seen from Sec. IV, both the vector

current $j_{a\mu}$ and the axial-vector current $j_{a\mu}^5$ for the spinor fields are conserved:

$$\partial^a j_{a\mu} = 0, \quad (5.1a)$$

$$\partial^a j_{a\mu}^5 = 0. \quad (5.1b)$$

It is obvious that the current also satisfies the massless Klein-Gordon equation

$$\square j_{a\mu} = 0. \quad (5.2)$$

This allows us to decompose the current into positive (containing creation operators) frequencies and negative (containing annihilation operators) frequencies in an invariant manner³³:

$$j_{a\mu} = j_{a\mu}^{(+)} + j_{a\mu}^{(-)}. \quad (5.3)$$

By choosing the light-cone variables u and v in the parameter space, we obtain from Eq. (5.1)

$$j_{+\mu} = j_{+\mu}(u), \quad (5.4a)$$

$$j_{-\mu} = j_{-\mu}(v). \quad (5.4b)$$

The commutation rules are those of Eq. (4.7):

$$[j_{+\mu}(u), j_{+\nu}(u')] = 2icg_{\mu\nu} \delta'(u - u'), \quad (5.5a)$$

$$[j_{-\mu}(v), j_{-\nu}(v')] = 2icg_{\mu\nu} \delta'(v - v'). \quad (5.5b)$$

This defines a quantum dynamical system for which we can introduce the conformal generators: the time translation H , the space translation P , the Lorentz boost M , and the dilatation operator D . These obey the algebra

$$[M, D] = [H, P] = 0, \quad (5.6a)$$

$$i[M, H] = i[D, P] = P, \quad (5.6b)$$

$$i[M, P] = i[D, H] = H, \quad (5.6c)$$

and are defined by

$$\begin{aligned} i[H, j_{+\nu}(u)] &= i[P, j_{+\nu}(u)] \\ &= g_{\nu\nu} \partial_u j_{+\nu}(u), \end{aligned} \quad (5.7a)$$

$$\begin{aligned} i[H, j_{-\nu}(v)] &= -i[P, j_{-\nu}(v)] \\ &= g_{\nu\nu} \partial_v j_{-\nu}(v), \end{aligned} \quad (5.7b)$$

$$\begin{aligned} i[D, j_{+\nu}(u)] &= i[M, j_{+\nu}] \\ &= g_{\nu\nu} (u \partial_u + 1) j_{+\nu} \\ &= g_{\nu\nu} \partial_u (u j_{+\nu}), \end{aligned} \quad (5.7c)$$

$$\begin{aligned} i[D, j_{-\nu}(v)] &= -i[M, j_{-\nu}] \\ &= g_{\nu\nu} (v \partial_v + 1) j_{-\nu} \\ &= g_{\nu\nu} \partial_v (v j_{-\nu}), \end{aligned} \quad (5.7d)$$

where the 1 on the right-hand side of the last two equations is the canonical dimension of the currents. The energy-momentum tensor is given by Eq. (4.15). It is divergenceless, symmetric, and traceless, so that

$$\Theta_+(u) = \Theta_{00} + \Theta_{01} = \frac{1}{2c} \sum_v g_{vv} : j_{+v}^2 :, \quad (5.8a)$$

$$\Theta_-(v) = \Theta_{00} - \Theta_{01} = \frac{1}{2c} \sum_v g_{vv} : j_{-v}^2 :, \quad (5.8b)$$

where the normal ordering is well defined due to Eq. (5.3). The conformal group generators can be defined in terms of the energy-momentum tensor as

$$H + P = \int \Theta_+(u) du, \quad (5.9a)$$

$$H - P = \int \Theta_-(v) dv, \quad (5.9b)$$

$$D + M = \int \Theta_+(u) u du, \quad (5.9c)$$

$$D - M = \int \Theta_-(v) v dv. \quad (5.9d)$$

In fact we can introduce the infinite conformal algebra²⁶:

$$L_n^+ = \frac{1}{2}i \int \Theta_+(u) u^{n+1} du, \quad (5.10a)$$

$$L_n^- = \frac{1}{2}i \int \Theta_-(v) v^{n+1} dv, \quad (5.10b)$$

so that Eq. (5.9) just gives $L_{-1}^{(\pm)}$ and $L_0^{(\pm)}$ only. From the commutation relation (and a similar commutator for Θ_-)

$$i[\Theta_+(u), \Theta_+(u')] = 2[\Theta_+(u) + \Theta_+(u')] \delta'(u - u') - \frac{2}{3}\pi \delta'''(u - u') \quad (5.11)$$

$$[\Theta_+(u), \psi_\mu(u', v', b_\mu)] = -g_{\mu\mu} \frac{a_\mu + \bar{a}_\mu \gamma_5}{c} : j_{+\mu}(u) \psi_\mu(u', v', b_\mu) : \delta(u - u') + g_{\mu\mu} \frac{i(a_\mu + \bar{a}_\mu \gamma_5)^2}{4\pi c} \psi_\mu(u', v', b_\mu) \delta'(u - u'), \quad (5.16a)$$

$$[\Theta_-(v), \psi_\mu(u', v', b_\mu)] = -g_{\mu\mu} \frac{a_\mu - \bar{a}_\mu \gamma_5}{c} : j_{-\mu}(v) \psi_\mu(u', v', b_\mu) : \delta(v - v') + g_{\mu\mu} \frac{i(a_\mu - \bar{a}_\mu \gamma_5)^2}{4\pi c} \psi_\mu(u', v', b_\mu) \delta'(v - v'). \quad (5.16b)$$

The equation of motion for the field is ($\mathcal{A} = A \cdot \gamma$)

$$i\partial\!\!\!/ \psi_\mu(x, b_\mu) = h_\mu : j_\mu(x) \psi_\mu(x, b_\mu) : g_{\mu\mu}, \quad (5.17)$$

where h_μ is the coupling constant. On the other hand, the space-time translations for the charged field ψ follow from Eqs. (5.9) and (5.16):

$$\begin{aligned} \partial_u \psi_\nu(u, v, b_\nu) &= \frac{1}{2}i[H + P, \psi_\nu(u, v, b_\nu)] \\ &= -\frac{i(a_\nu + \bar{a}_\nu \gamma_5)}{2c} g_{\nu\nu} : j_{+\nu}(u) \psi_\nu(u, v, b_\nu) :, \end{aligned} \quad (5.18a)$$

we obtain the conformal algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{m+n,0} \frac{1}{3}n(n^2 - 1), \quad (5.12)$$

which is identical to the Virasoro algebra, Eq. (2.7).

Let us now introduce the spinor fields with different spins.³² Each spinor field has a definite spin, designated by the quantity b , to be defined later. Following Johnson,³¹ we postulate the commutation rules between the currents and the fields (repeated index not summed):

$$[j_{+\mu}(u), \psi_\nu(u', v', b_\nu)] = -g_{\mu\nu} (a_\nu + \bar{a}_\nu \gamma_5) \psi_\nu \delta(u - u'), \quad (5.13a)$$

$$[j_{-\mu}(u), \psi_\nu(u', v', b_\nu)] = -g_{\mu\nu} (a_\nu - \bar{a}_\nu \gamma_5) \psi_\nu \delta(v - v'), \quad (5.13b)$$

where we choose the set of γ matrices [Eq. (4.11)] in which γ_5 is diagonal. $a_\nu, \bar{a}_\nu$ are constants to be determined. Following Klaiber,³³ we define the normal ordering for a product of a current with a spinor field by

$$: j_a \psi : = j_a^{(+)} \psi + \psi j_a^{(-)}. \quad (5.14)$$

In fact we can take Eq. (5.13) to be true for $j_a^{(+)}$ and $j_a^{(-)}$ separately, using the following identity:

$$\begin{aligned} \delta(x) &= \delta^{(+)}(x) + \delta^{(-)}(x) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right). \end{aligned} \quad (5.15)$$

Now it is straightforward to evaluate the following commutators:

$$\begin{aligned} \partial_v \psi_\nu(u, v, b_\nu) &= \frac{1}{2}i[H + P, \psi_\nu(u, v, b_\nu)] \\ &= \frac{-i(a_\nu - \bar{a}_\nu \gamma_5)}{2c} g_{\nu\nu} : j_{-\nu}(v) \psi_\nu(u, v, b_\nu) :. \end{aligned} \quad (5.18b)$$

Equations (5.17) and (5.18) are consistent provided

$$a_\nu - \bar{a}_\nu = h_\nu c. \quad (5.19)$$

But we also obtain two more equations,¹⁷ namely those for $\partial_u \psi_1$ and $\partial_v \psi_2$. They provide additional information which is introduced by DFZ to replace the need of the method of point separation in the

equation of motion (5.17).

We can now look at the Lorentz and the dilatation properties of $\psi_\mu(u, v, b_\mu)$. From Eqs. (5.9), (5.16), and (5.18),

$$i[M, \psi_\nu(u, v, b_\nu)] = \left(u\partial_u - v\partial_v + g_{\nu\nu} \frac{a_\nu \bar{a}_\nu}{2\pi c} \gamma_5 \right) \times \psi_\nu(u, v, b_\nu), \quad (5.20)$$

$$i[D, \psi_\nu(u, v, b_\nu)] = \left(u\partial_u + v\partial_v + g_{\nu\nu} \frac{a_\nu^2 + \bar{a}_\nu^2}{4\pi c} \right) \times \psi_\nu(u, v, b_\nu), \quad (5.21)$$

so the spin s_ν and the anomalous dimension d_ν of ψ_ν are given by³²

$$s_\nu = \frac{a_\nu \bar{a}_\nu}{2\pi c} g_{\nu\nu}, \quad (5.22)$$

$$\begin{aligned} d_\nu &= \frac{a_\nu^2 + \bar{a}_\nu^2}{4\pi c} g_{\nu\nu} \\ &= s_\nu + \frac{(a_\nu - \bar{a}_\nu)^2}{4\pi c} g_{\nu\nu} \\ &= s_\nu + \frac{h_\nu^2 c}{4\pi} g_{\nu\nu}, \end{aligned} \quad (5.23)$$

where we have used the relation (5.19). Under a Lorentz transformation given by a "boost" angle θ such that $u \rightarrow e^{-\theta} u$ and $v \rightarrow e^\theta v$, we have

$$\begin{aligned} j_+(u) &\rightarrow e^{\theta} j_+(e^\theta u), \\ j_-(v) &\rightarrow e^{-\theta} j_-(e^{-\theta} v), \\ \psi_\nu(u, v, b_\nu) &\rightarrow e^{\gamma_5 \theta s_\nu} \psi_\nu(e^\theta u, e^{-\theta} v, b_\nu). \end{aligned} \quad (5.24)$$

The equations of motion (5.17) permit any transformation law for ψ . It is crucial for our purpose to have ψ for any spin s . To do so, we have introduced from the beginning an extra label b_ν associated with each ψ_ν , having the following definition:

$$b_\nu = -\frac{1}{(2\pi c)^{1/2}} \left(\frac{a_\nu + \bar{a}_\nu}{2} \right). \quad (5.25)$$

It follows from Eqs. (5.19), (5.22), and (5.23) that

$$s_\nu = b_\nu^2 g_{\nu\nu} - \frac{h_\nu^2 c}{8\pi} g_{\nu\nu}, \quad (5.26a)$$

$$d_\nu = b_\nu^2 g_{\nu\nu} + \frac{h_\nu^2 c}{8\pi} g_{\nu\nu}. \quad (5.26b)$$

Given h_ν , the spin and the anomalous dimension

$$\begin{aligned} F &= \exp\{i(2\pi/c)^{1/2} [bJ_+^{(+)}(u) - b'J_+^{(+)}(u')] - \frac{1}{2}i\hbar[J_-^{(+)}(v) - J_-^{(+)}(v')]\} \psi_1(u, v, b) \psi_1^\dagger(u', v', b') \\ &\times \exp\{i(2\pi/c)^{1/2} [bJ_+^{(-)}(u) - b'J_+^{(-)}(u')] - \frac{1}{2}i\hbar[J_-^{(-)}(v) - J_-^{(-)}(v')]\}, \end{aligned} \quad (5.30)$$

where $J^{(\pm)}$ is the $(\pm)$ part of J (see Sec. VI for a more precise definition) in correspondence to the $j^{(\pm)}$ decomposition [Eq. (5.3)]. Using the commutation relations between currents and fields,

of ψ_ν are determined by b_ν . This means we are studying an extended Thirring model where we have fields ψ_ν with any spin s_ν but only interacting with themselves. Formally, the Lagrangian has the following form:

$$\mathcal{L} = \sum_{\nu, b_\nu} \bar{\psi}_\nu \not{\partial} \psi_\nu + g_{\nu\nu} h_\nu (\bar{\psi}_\nu \gamma^a \psi_\nu) (\bar{\psi}_\nu \gamma_a \psi_\nu). \quad (5.27)$$

We emphasize that if ψ_ν has a nonzero mass, the resulting equation of motion will determine the spin of the spinor field to be $\frac{1}{2}$. Also, because the parameter space is two-dimensional, there is only one Lorentz generator and so the Lorentz transformation can be trivially satisfied by any spin.

We have introduced three sets of "constants," namely $(a_\nu, \bar{a}_\nu)$, (s_ν, d_ν) , and (b_ν, h_ν) , such that any two sets are determined when the third is given. For any given Lagrangian, the coupling constant h_ν is fixed; for any given $\psi_\nu(x, b_\nu)$, b_ν is fixed. In general, the values of a_ν , $\bar{a}_\nu$, s_ν , and d_ν are dependent on $\psi_\nu(x, b_\nu)$.

With the definition

$$j_{+\nu}(u) = \partial_u J_{+\nu}(u), \quad (5.28a)$$

$$j_{-\nu}(v) = \partial_v J_{-\nu}(v), \quad (5.28b)$$

we can integrate Eq. (5.18) to obtain (where both the Greek indices and the metric $g_{\mu\nu}$ are suppressed to simplify the equation)

$$\begin{aligned} \psi(u, v, b) &= : \exp \left\{ -\frac{i}{2c} (a + \bar{a} \gamma_5) [J_+(u) - J_+(u_0)] \right\} \\ &\times \exp \left\{ -\frac{i}{2c} (a - \bar{a} \gamma_5) [J_-(v) - J_-(v_0)] \right\} \\ &\times \psi(u_0, v_0, b) : . \end{aligned} \quad (5.29)$$

Because of the charge selection rules, all non-vanishing Wightman functions contain equal numbers of ψ_1 's and $\psi_1^\dagger$'s (and ψ_2 's and $\psi_2^\dagger$'s). Therefore, all we need to do is to commute the fields so that they are grouped in pairs and then evaluate the pairs in terms of currents. Since commutation relations of the currents are given, the Wightman functions (or equivalently the Green's function) can be evaluated explicitly.

To evaluate the product $\psi_1(u, v, b) \psi_1^\dagger(u', v', b')$ we consider the following expression:

it can be shown that F is a c -number function $F = F(u - u', v - v')$. Applying the equations of motion (5.18) directly to Eq. (5.30), a pair of first-order differential equations involving F is obtained.

Putting the solution into Eq. (5.30), we finally obtain

$$\begin{aligned} \psi_1(u, v, b) \psi_1^\dagger(u', v', b') &= f_0 [i(u - u') + \epsilon]^{-2bb'} \\ &\times [i(v - v') + \epsilon]^{-h^2 c/4\pi} : \exp\{i(2\pi/c)^{1/2} [bJ_+(u) - b'J_+(u')] - \frac{1}{2}ih[J_-(v) - J_-(v')]\} : . \end{aligned} \quad (5.31a)$$

Similarly we can derive

$$\begin{aligned} \psi_1^\dagger(u', v', b') \psi_1(u, v, b) &= f_0 [i(u' - u) + \epsilon]^{-2bb'} \\ &\times [i(v' - v) + \epsilon]^{-h^2 c/4\pi} : \exp\{i(2\pi/c)^{1/2} [bJ_+(u) - b'J_+(u')] - \frac{1}{2}ih[J_-(v) - J_-(v')]\} : , \end{aligned} \quad (5.31b)$$

where f_0 is a positive real number whose value determines the normalization of the fields. ϵ is, by definition, an infinitesimal positive number. A similar set of equations can be derived for ψ_2 (obtainable from the above equations by $\bar{a} \rightarrow -\bar{a}$). Equation (5.31) is a very simple and nice example of both an exact short-distance expansion, as first postulated by Wilson,³⁴ and an exact light-cone expansion, as suggested by Frishman³⁵ and Brandt and Preparata.³⁵ In Sec. VI we shall see that the light-cone singularities are closely associated with the resonance poles in the DRM, and the short-distance singularities with the resonance poles in the Virasoro-Shapiro model.

Let us remark that the currents can also be defined in terms of the fields such that they still obey the canonical commutation relations. We refer the reader to DFZ for a discussion on this point.

From Eq. (5.31) we obtain immediately the expression

$$\begin{aligned} \psi_{1\nu}(u, v, b) \psi_{1\nu}^\dagger(u', v', b') \\ = \exp[-2\pi i b_\nu b'_\nu g_{\nu\nu} \eta(u - u') - \frac{1}{2}ih_\nu^2 g_{\nu\nu} \eta(v - v')] \\ \times \psi_{1\nu}^\dagger(u', v', b') \psi_{1\nu}(u, v, b), \end{aligned} \quad (5.32)$$

where $\eta(\omega) = \omega/|\omega|$. Let us consider the case when the fields have the same spin (i.e., $b_\nu = b'_\nu$). For spacelike separation, $(u - u') \cdot (v - v') < 0$, the above expression becomes

$$\begin{aligned} \psi_{1\nu}(u, v, b_\nu) \psi_{1\nu}^\dagger(u', v', b_\nu) \\ - (-1)^{2s_\nu \eta(u - u')} \epsilon_{\nu\nu} \psi_{1\nu}^\dagger(u', v', b_\nu) \psi_{1\nu}(u, v, b_\nu) = 0, \end{aligned} \quad (5.33)$$

so that it vanishes as a commutator for integer spin s_ν and vanishes as an anticommutator for half-integer spin. Equation (5.31) generalizes the connection between spin and statistics to fields with any spin, and also to fields with different spins. In general, we have "funny" statistics. $\psi_{1\nu}$ and $\psi_{1\mu}^\dagger$ always commute when $\mu \neq \nu$.

VI. THE THIRRING MODEL AND THE DRM

In this section we present the conditions under which the vacuum expectation values of specific

products of the fields of the extended Thirring model become the integrands of the DRM. We first postulate an expression for the physical boson field in terms of a product of Thirring fields which are considered as some sort of underlying constituents. The simplest possible expression is obtained by borrowing the Harari-Rosner dual quark picture, where a meson is a quark-antiquark pair. However, here, the Thirring fields obey "funny" statistics and are not identified as physical quarks, but rather as mathematical constituents that build up the physical mesons. Scattering takes place when these constituents are exchanged by physical meson fields, just like conventional dual quark diagrams (e.g., see Fig. 1). The connection with the DRM is then presented, following the application of the duality postulates.

Let us begin by quantizing the Thirring currents. Q_μ in the DRM [Eq. (2.3)] obeys the zero-mass Klein-Gordon equation:

$$\square Q_\mu(z) = \partial_z \partial_{\bar{z}} Q_\mu(z) = 0. \quad (6.1)$$

For equal r , we have the following identity ($z = r e^{i\theta}$):

$$\delta(\theta - \theta') = iz \delta(z - z') \delta_{rr'}. \quad (6.2)$$

This allows us to write Eq. (2.5) as

$$\left[Q_\mu(z), \frac{P_\nu(z')}{z'} \right]_{r=r'} = 2\pi i g_{\mu\nu} \delta(z - z'), \quad (6.3a)$$

$$\left[\frac{P_\mu(z)}{z}, \frac{P_\nu(z')}{z'} \right]_{r=r'} = 2\pi i g_{\mu\nu} \delta'(z - z'). \quad (6.3b)$$

Comparing them with the commutation relations of the Thirring currents (5.4) and (5.28), we have

$$[j_{+\mu}(u), j_{+\nu}(u')] = 2ic g_{\mu\nu} \delta'(u - u'), \quad (6.4a)$$

$$[j_{+\mu}(u), J_{+\nu}(u')] = -2ic g_{\mu\nu} \delta(u - u'), \quad (6.4b)$$

which are obviously valid for $|r| = |r'|$ since they are independent of v , and we can make the following identification (note that $\square J_{+\nu} = \partial_u \partial_v J_{+\nu} = 0$):

$$j_{+\mu}(z) = \left(\frac{c}{\pi} \right)^{1/2} \frac{P_\mu(z)}{z}, \quad (6.5a)$$

$$J_{+\mu}(z) = \left(\frac{c}{\pi}\right)^{1/2} Q_{\mu}(z), \quad (6.5b)$$

where Q_{μ} and P_{μ} are quantized already [Eqs. (2.4) and (2.5)]. Note that when the timelike coordinate in the parameter space is rotated, $u = z = re^{i\theta}$, $-v = \bar{z} = re^{-i\theta}$, one can verify explicitly that Eqs. (6.4) are satisfied. The $j_{-}(v)$ current can be quan-

tized in the same way.

For the purpose of this work, we are interested only in the product³⁶

$$\prod_{\mu=0,1,2,3} \psi_{\mu}(u, v, b_{\mu}) = \Psi(u, v, b).$$

A physical meson field is postulated to be a product of Thirring fields having the following form:

$$\begin{aligned} \lim_{\substack{u \rightarrow u' \\ v \rightarrow v'}} c'' \Psi(u, v, b) \Psi^{\dagger}(u', v', b') &= \lim_{\substack{u \rightarrow u' \\ v \rightarrow v'}} c' [i(u - u') + \epsilon]^{-2b \cdot b'} [i(v - v') + \epsilon]^{-h \cdot h c / 4\pi} \\ &\quad \times : \exp[i\sqrt{2} [b \cdot Q(u) - b' \cdot Q(u')] - \frac{1}{2} i h \cdot [J_{-}(v) - J_{-}(v')]] : \\ &= \lim_{\substack{u \rightarrow u' \\ v \rightarrow v'}} c' [i(u - u') + \epsilon]^{-2b \cdot b'} [i(v - v') + \epsilon]^{-h \cdot h c / 4\pi} : \exp[i\sqrt{2} (b - b') \cdot Q(u)] : , \quad (6.6) \end{aligned}$$

where

$$A \cdot B = \sum_{\mu} g_{\mu\mu} A_{\mu} B_{\mu}$$

and c'', c' are constants. This is motivated by the dual quark picture where $\Psi(u, v, b)$ is treated as a quarklike constituent and $\Psi^{\dagger}(u', v', b')$ is treated as an antiquarklike constituent. Introducing the Regge slope α' , the second duality postulate becomes

$$(\alpha')^{1/2} k_{\mu} = b_{\mu} - b'_{\mu} \quad (6.7)$$

so that $b_{\mu}/(\alpha')^{1/2}$ becomes the momentum of the constituent field. The relation (6.7) is a stronger and more precise statement than the original duality postulate. Recalling the relation (5.26) between the dimension and the spin in the parameter space,

we observe that the dimension of the product of fields $\Psi(u, v, b) \Psi^{\dagger}(u', v', b')$ is

$$(b - b') \cdot (b - b') = (b - b')^2 = \alpha' k^2 = -\alpha' m^2, \quad (6.8)$$

where m is the mass of the physical meson field. The product of $\prod_{\mu} \psi_{\mu}$ is necessary in order to introduce all components of the momentum of a physical field.

At this point, we would like to recall from earlier discussions of the string that the vacuum state in the parameter space is also the vacuum state in the physical space ($j^{(-)}|0\rangle = 0$).

The dual quark diagrams suggest that the scattering of mesons takes place by exchanging their constituents. Thus we are led to study the following Thirring-Wightman function in the parameter space in connection with the four-point scattering amplitude (see Fig. 1):

$$\begin{aligned} \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \langle 0 | \Psi(u_1, v_1, b_1) \Psi^{\dagger}(u'_1, v'_1, b'_1) \Psi(u_2, v_2, b_2) \Psi^{\dagger}(u'_2, v'_2, b'_2) \Psi(u_3, v_3, b_3) \Psi^{\dagger}(u'_3, v'_3, b'_3) \Psi(u_4, v_4, b_4) \Psi^{\dagger}(u'_4, v'_4, b'_4) | 0 \rangle \\ = f' \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} (i\Delta u + \epsilon)^{-2(b_1 \cdot b_2 + b_2 \cdot b_3 + b_3 \cdot b_4 + b_4 \cdot b_1)} (i\Delta v + \epsilon)^{-h \cdot h c / \pi} \left\langle 0 \left| \prod_{j=1}^4 : \exp[i\sqrt{2} (b_j - b_{j+1}) \cdot Q(u_j)] : \right| 0 \right\rangle , \quad (6.9) \end{aligned}$$

where $\Delta u = u_j - u'_j$, $\Delta v = v_j - v'_j$, and f' is a constant. Here we have four meson momenta

$$k_i = \frac{1}{\sqrt{\alpha'}} (b_i - b_{i+1})$$

and four constituent momenta $b_i/(\alpha')^{1/2}$. However, the sum of the meson momenta is automatically zero (i.e., conservation of momentum). This provides us the extra freedom of choosing the constituent fields such that

$$b_1 \cdot b_2 + b_2 \cdot b_3 + b_3 \cdot b_4 + b_4 \cdot b_1 = 0$$

without changing any of the physically measurable momenta. We can also fix $h \cdot h = 0$ ($h_{\mu} \neq 0$ in gen-

FIG. 1. The dual-quark-like diagram for the four-point function.

eral) so that Eq. (6.9) has the following simplified expression:

$$f' \left\langle 0 \left| \prod_{j=1}^4 : \exp[i\sqrt{2}(b_j - b_{j+1}) \cdot Q(u_j)] : \right| 0 \right\rangle. \quad (6.10)$$

Let us now examine the exact meaning of the normal ordering of the exponential factors. From Eqs. (5.10), (5.16), and (5.25), it is straightforward to derive the following commutation relation:

$$[L_n, \Psi(u, v, b)] = u^{n+1} \partial_u \Psi + b \cdot b(n+1) u^n \Psi. \quad (6.11)$$

From this we obtain³⁷

$$\begin{aligned} [L_n, \Psi(u, v, b) \Psi^\dagger(u', v', b')] &= \left[L_n, \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \Psi(u, v, b) \Psi^\dagger(u', v', b') \right] \\ &= u^{n+1} \partial_u (\Psi \Psi^\dagger) + (b - b')^2 (n+1) u^n (\Psi \Psi^\dagger), \end{aligned} \quad (6.12)$$

where $\Delta u, \Delta v$ are independent of u, v . Comparing this with Eqs. (2.4) and (2.8), we have, for $\alpha' k^2 = 1$,

$$\begin{aligned} \frac{V(u, k)}{u} &= : \exp[i\sqrt{2\alpha'} k \cdot Q(u)] : \\ &= \exp[i(2\alpha')^{1/2} k \cdot Q^{(+)}(u)] \exp[i(2\alpha')^{1/2} k \cdot q] \\ &\quad \times \exp[(2\alpha')^{1/2} k \cdot p \ln u] \exp[i(2\alpha')^{1/2} k \cdot Q^{(-)}(u)]. \end{aligned} \quad (6.13)$$

This definition for normal ordering is slightly different from that in the traditional DRM. Note that from Eq. (6.5) we have

$$j^{(-)}(u) = -\left(\frac{c}{\pi}\right)^{1/2} \left(\frac{i\dot{p}}{u} + \sum_{n=1}^{\infty} a_n u^{-n-1} \right) \quad (6.14)$$

and

$$j^{(+)}(u) = \left(\frac{c}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} a_n^\dagger u^{n-1}.$$

For the rest of our discussions, we adopt the definition of normal ordering provided by Eq. (6.13).

To obtain the "Thirring Green's function" all we need is to introduce a "time ordering" for the fields according to the proper time τ_j or equivalently $|u_j|$. For example,

$$\begin{aligned} R[\Psi(u) \Psi(u')] &= \theta(|u| - |u'|) \Psi(u) \Psi(u') \\ &\quad + \theta(|u'| - |u|) \Psi(u') \Psi(u). \end{aligned} \quad (6.15)$$

Thus we have a complete identification of the Thirring Green's function in the parameter space with

$$\lim_{\Delta u, \Delta v \rightarrow 0} \langle 0 | \Psi(u_1, v_1, b_1) \Psi^\dagger(u'_1, v'_1, b'_1) \Psi(u_2, v_2, b_2) \cdots \Psi(u_N, v_N, b_N) \Psi^\dagger(u'_N, v'_N, b'_N) | 0 \rangle$$

the "dual Green's function" (i.e., the integrand in the DRM) in the physical space.

Let us now derive the scattering amplitude of four mesons from the dual Green's function. The coupling constants are dimensionless. Therefore the Thirring model has no dimensional scales and it is impossible to define meaningfully an asymptotic region. This means that one can construct Wightman and Green's functions in the parameter space but not S-matrix elements.³⁸ In states and out states cannot be defined properly. However, in going over to the physical space via the duality postulates, the introduction of the Regge slope α' provides a needed dimensional parameter. This permits us to define the asymptotic regions and hence the S-matrix elements. As a consequence of the FHJ proper-time quantization, the asymptotic domains are at $r = |u| \rightarrow 0, \infty$.

For the four-point function of particles $\alpha_1, \alpha_2, \alpha_3$, and α_4 we have $\langle \alpha_1 \alpha_2 \text{ out} | \alpha_3 \alpha_4 \text{ in} \rangle$. For simplicity, let us take α_i to be the scalar ground-state particles. The scattering amplitude is a function of the scalar products of momenta. Since the Lagrangian is conformally invariant, one may want the scattering amplitude to reflect this symmetry. However, the vacuum state breaks the infinite conformal algebra (5.10), but remains invariant only under the projective transformations (i.e., $L_{\pm 1}$ and L_0 , which form a closed algebra). Therefore we demand the scattering amplitudes to be projective-invariant only. Neglecting the non-scattering part of the amplitude, the Veneziano amplitude emerges, relating the scattering amplitude to the vacuum expectation value (6.10),

$$B_4 = \int \frac{\prod_{i=1}^4 du_i}{d\Omega} \left\langle 0 \left| R \left[\prod_{i=1}^4 \frac{V(u_i, k_i)}{u_i} \right] \right| 0 \right\rangle, \quad (6.16)$$

where the invariant differential volume is given by Eq. (2.2) with $z_i = u_i$. R is the time ordering of the u_i 's. For the complete amplitude, we have to include all terms with different u orderings. Because of the projective invariance property, there are only three independent terms for the four-point functions.

The generalization of the connection between the extended Thirring model and the DRM to an N -point function is straightforward. To summarize, the appropriate vacuum expectation values in the parameter space become the vacuum expectation values in the physical space via the duality postulates:

$$\rightarrow \left\langle 0 \left| \prod_{j=1}^N : \exp[i\sqrt{2}(b_j - b_{j+1}) \cdot Q(u_j)] : \right| 0 \right\rangle.$$

Projective invariance fixes the scattering amplitudes in terms of the vacuum expectation values. We emphasize again that the S matrix does not exist in the parameter space but is defined in the physical space. As is clear, the scattering amplitude for four spinning particles can also be derived, using the projection formulas of Refs. 21 and 22.

The coupling constant of the DRM is not determined. The normalization of the scattering amplitude comes from the following places: (1) the normalization of the Thirring fields f_0 , (2) the normalization of the physical hadron field c' , (3) $(i\Delta u + \epsilon)^{2b \cdot b'}(i\Delta v + \epsilon)^{-h^2 c/4\pi}$, (4) the infinite v_i in-

tegrations, (5) possible normalization factors from the "reduction formulas", and (6) possible phase factors from commuting Thirring fields (see below). In earlier discussions, we impose constraints [discussions following Eq. (6.9)] such that the infinite normalization due to (3) becomes finite (equal to unity). Actually we might as well relax these constraints and combine all the normalization factors to give the coupling constant of the DRM.

The connection between the light-cone singularity in the parameter space and the resonance poles in the DRM should be apparent now. For simplicity, we set $b_2 \cdot b_3 + b_3 \cdot b_4 = 0$ in the expression

$$\begin{aligned} & [\Psi(u, v, b_2)\Psi^\dagger(u, v, b_3)][\Psi(u', v', b_3)\Psi^\dagger(u', v', b_4)] \\ &= (u - u')^{2(b_2 \cdot b_3) \cdot (b_3 \cdot b_4)} : \exp[\sqrt{2} i(b_2 - b_3) \cdot Q(u) + \sqrt{2} i(b_3 - b_4) \cdot Q(u')] : . \end{aligned} \quad (6.17)$$

With $(\alpha')^{1/2} k_2 = b_2 - b_3$ and $(\alpha')^{1/2} k_3 = b_3 - b_4$ the strength of the singularity as $u \rightarrow u'$ is determined by $2\alpha' k_2 \cdot k_3$ only and independent of the Thirring coupling constant h . Taking the expectation value of the above between

$$\langle 0 | \exp[i(2\alpha')^{1/2} k_1 \cdot q]$$

and

$$\exp[i(2\alpha')^{1/2} k_4 \cdot q] | 0 \rangle ,$$

we obtain

$$(u - u')^{2\alpha' k_2 \cdot k_3 u'^2 \alpha' k_2 \cdot k_4 u'^2 \alpha' k_3 \cdot k_4} . \quad (6.18)$$

It is well known in the DRM that the singularity from $u \rightarrow u'$ in the above expression survives to be the resonance pole in the scattering amplitude only when $2\alpha' k_2 \cdot k_3$ is a positive integer.

It should be noted that, throughout, we keep h_μ nonzero so that the Thirring interactions are present. However, if we let h_μ go to zero, the whole formalism remains intact; that is to say, we still have scattering in the physical space. This is because the interaction in the physical space does not come from the Thirring interaction, but rather from the exchange of the Thirring constituent fields, as in dual quark diagrams. The physical hadron fields arise when the space-time position of the constituent fields approach each other in the parameter space.

So far we have neglected the other component of the Thirring field ψ_2 . Introducing ψ_2 brings in the current $j_-(v)$. The inclusion of both ψ_1 and ψ_2 can give rise to the Virasoro-Shapiro model, following essentially the same path developed in this section [see Eqs. (2.17)–(2.20)]. Defining

$$j_-(v) = \frac{d}{dv} J_-(v) , \quad (6.19)$$

we make the following identification:

$$J_{-\mu}(\bar{z}) = \left(\frac{c}{\pi}\right)^{1/2} Q_\mu(\bar{z}) . \quad (6.20)$$

The hadron field becomes $V(z, k)V(\bar{z}, k)/z\bar{z}$ and the ordering is in $r = (z\bar{z})^{1/2}$. The resonance poles in the model emerge from the short-distance singularities, namely $z \rightarrow z'$ and $\bar{z} \rightarrow \bar{z}'$, as compared with the case in the DRM. Examining Eqs. (5.9)–(5.12), we see that the evolution operator in the DRM is actually the dilatation operator plus the boost operator, $L_0 = \frac{1}{2}i(D + M)$, while in the Virasoro-Shapiro model it is the dilatation operator $D = -i(L_0 + \bar{L}_0)$ itself that generates the evolution in the proper time r in the parameter space.

We remark in passing that one way to obtain the Neveu-Schwarz model is by introducing another set of free spinor fields to the Thirring model.

In this section we assume a definite constituent structure for the physical meson field. In the next section we examine this assumption by studying a couple of examples with different constituent structures.

VII. THE STRUCTURE OF HADRON FIELDS

Our main purpose in this section is to present evidence that the physical hadron field is relatively independent of its detailed constituent structure. As our first and almost trivial example, let us consider the following product of fields:

$$\begin{aligned} & \Psi(u, v, b_1)\Psi(u, v, b_3)\Psi^\dagger(u, v, b_2)\Psi^\dagger(u, v, b_4) \\ &= \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon' \rightarrow 0}} (i\epsilon)^{-2b_1 \cdot b_2 - 2b_3 \cdot b_4} \epsilon'^{2(b_1 - b_2) \cdot (b_3 - b_4)} \\ & \times \frac{V(u, b_1 - b_2 + b_3 - b_4)}{u} . \end{aligned} \quad (7.1)$$

If the momentum of the physical hadron field is $(\alpha')^{1/2} k = b_1 - b_2 + b_3 - b_4$, we have, within a normalization factor, the vertex operator in the DRM. The corresponding four-point duality diagram is shown in Fig. 2. In general, the meson field can

be constructed from equal numbers of quarklike and antiquarklike constituents. For completeness, let us derive the commutation relation of the fields themselves. From Sec. V and Eq. (2.12), we have the following commutator (for $\hbar^2 = 0$):

$$[\Psi(u, v, b_1) \Psi^\dagger(u, v, b_2)] [\Psi(u', v', b_3) \Psi^\dagger(u', v', b_4)] = \exp[2\pi i \eta(u - u')(b_1 - b_2) \cdot (b_3 - b_4)] \\ \times [\Psi(u', v', b_3) \Psi^\dagger(u', v', b_4)] [\Psi(u, v, b_1) \Psi^\dagger(u, v, b_2)]. \quad (7.2)$$

Using this together with Eq. (5.31), we may deduce directly the commutation relation

$$\Psi(u, v, b) \Psi(u', v', b') = \exp[2\pi i b \cdot b' \eta(u - u')] \Psi(u', v', b') \Psi(u, v, b). \quad (7.3)$$

This means all fields commute when they are at the same point.

Next, let us study a physical hadron field which is different from the vertex operator in the DRM. For definiteness, we consider a physical hadron field which has the form

$$W(u, v, k) = \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0}} \Psi(u, v, b) \Psi(u', v', b') \Psi(u'', v'', b'') \quad (7.4)$$

such that $(\alpha')^{1/2} k = b + b' + b''$. For $k_\mu \neq 0$, W is also

a function of $J_-(v)$ and hence is different from the physical hadron field discussed earlier. However, it has identical transformation properties under L_n :

$$[L_n, W(u, v, k)] = u^{n+1} \partial_u W(u, v, k) \\ + (n+1) \alpha' k^2 u^n W(u, v, k), \quad (7.5)$$

where Eqs. (5.16) and (6.11) have been used. To study this vertex further, let us consider the process shown in Fig. 3:

$$\langle 0 | V(u_1, v_1, k_1) W(u_2, v_2, k_2) W^\dagger(u_3, v_3, k_3) V(u_4, v_4, k_4) | 0 \rangle \\ = \langle 0 | [\Psi(u_1, v_1, b_1) \Psi^\dagger(u_1, v_1, b_2)] [\Psi(u_2, v_2, b_2) \Psi(u_2, v_2, b_3) \Psi(u_2, v_2, b_4)] \\ \times [\Psi^\dagger(u_3, v_3, b_4) \Psi^\dagger(u_3, v_3, b_3) \Psi^\dagger(u_3, v_3, b_5)] [\Psi(u_4, v_4, b_5) \Psi^\dagger(u_4, v_4, b_1)] | 0 \rangle, \quad (7.6)$$

where

$$\begin{aligned} (\alpha')^{1/2} k_1 &= b_1 - b_2, \\ (\alpha')^{1/2} k_2 &= b_2 + b_3 + b_4, \\ (\alpha')^{1/2} k_3 &= -b_3 - b_4 - b_5, \\ (\alpha')^{1/2} k_4 &= b_5 - b_1, \end{aligned} \quad (7.7)$$

FIG. 2. An alternate dual-quark-like diagram for the four-point function with the same vertex operator, $x_i = (u_i, v_i)$.

FIG. 3. The (s, t) -channel term of the four-point function with different vertex operators.

with the ordering ($u_j > u_{j+1}$), we obtain

$$(1 - u_3)^{2\alpha' k_2 \cdot k_3} u_3^{2\alpha' k_3 \cdot k_4}, \quad (7.8)$$

which gives the same Veneziano scattering amplitude. This means we can pick a different constituent structure for the physical hadron field without changing any of the measurable quantities. In fact, along exactly the same line, the scattering amplitude corresponding to Fig. 4 is calculated to be the (s, u) term of the four-point function [where Eq. (7.8) gives the (s, t) term].

The relative independence of the hadron fields on the specific details of their underlying constituent structures is, we believe, a consequence of the emphasis placed on the currents as observables instead of the fields, and a consequence of the basic exchange mechanism that is responsible to the scattering in the physical space. We remark that a Lorentz transformation in the physical space corresponds to a transformation of one set of fields to another set of fields in the parameter space. The picture obtained above is closer in spirit to the parton ideas than to the quark model in the sense that, in scattering, the hadrons can exchange an unspecified number of constituents, constrained only by the conservation of momenta. To be more accurate, the dual quark diagrams we draw in this work should be called dual constituent diagrams.

VIII. REMARKS

Here we concern ourselves with two topics only: the covariant versus the noncovariant quantizations of a relativistic string, and the meaning of the scale dimension.

A. Covariant vs noncovariant quantizations of a relativistic string

With the introduction of current j_a in Eq. (4.4), we can write the free-string Lagrangian in Eq. (3.1) as (μ, ν summed)

$$\mathcal{L} = [-j_\sigma^\mu j_{\sigma\mu} j_\tau^\nu j_{\tau\nu} + (j_\sigma^\mu j_{\tau\mu})^2]^{1/2}, \quad (8.1)$$

with the energy-momentum tensor vanishing as a consequence, or we can write the free-string Lagrangian (3.3) as

$$\mathcal{L} = \frac{1}{2} j_a^\mu j_\mu^a, \quad (8.2)$$

with the energy-momentum tensor vanishing as a constraint. Our conviction is to treat the currents as fundamental entities upon which the complete theory can be based. If we introduce the boson field [via Eq. (4.4)] we get back, of course, the free-string picture. However, if we choose the spinor field representation $j_a = \bar{\psi} \gamma_a \psi$, the above free-string Lagrangian (8.2) becomes the inter-

FIG. 4. The (s, u) -channel term of the four-point function with different vertex operators.

action term in the Thirring Lagrangian; we learn from Sec. VI that a product (a Lorentz scalar in the physical space) of the Thirring spinor fields gives precisely the vertex operator in the dual model. This suggests that, somehow, the free relativistic string has a natural built-in interaction to it.

In the covariant quantization of the free string (or equivalently, the study of the DRM spectrum), the interaction vertex operator has to be introduced; it is from the commutation relation between the energy-momentum tensor (or equivalently, the Virasoro algebra) and the vertex operator that the condition $\alpha_0 = 1$ emerges. The no-ghost proof of the free string (or equivalently, the DRM spectrum) relies heavily on the properties of the vertex operator. When the interaction is turned on in the DRM, the spectrum remains unchanged.

In the noncovariant quantization procedure, no knowledge of the interaction is required. The $\alpha_0 = 1$ condition emerges rather from the Lorentz properties of the quantization. The obvious advantage of this approach is that no knowledge of the interaction is required. However, the quantization is possible only in 26 dimensions. We believe this is due to the fact that the quantum string in the physical space has extra modes which do not have a classical analog.

The classical vibrating string has only two sets of independent transverse modes. In the noncovariant quantization only the independent variables are quantized, so that the quantum string has the same number of degrees of freedom as the classical string.

In the covariant quantization all variables are quantized. Constraints among the variables are then imposed weakly. All modes (i.e., states) that can satisfy the constraints are present. Because of the presence of the c -number Schwinger term in the commutator of the energy-momentum tensor (i.e., the c number in the Virasoro algebra) we find that there exist extra longitudinal modes which also satisfy the constraints and hence can

be present. The presence of the c number is due to normal ordering, a pure quantum-mechanical effect that has no classical analog. This allows us to interpret the quantum string as having one more degree of freedom than the classical string. The value of the Schwinger term depends on the number of space-time dimensions. If we go to 26 dimensions, this extra degree of freedom disappears as the longitudinal modes effectively decouple. Only in this case do the quantum and the classical strings have the same number of degrees of freedom, allowing us to quantize the string noncovariantly. This means that for a system with more degrees of freedom in the quantized version than that in the classical version, canonical noncovariant quantization is inadequate. In this paper we have resorted to covariant quantization. However, one can also carry out the noncovariant quantization by introducing noncanonical operators into the procedure.³⁹

There is another difference between the string and Thirring formalisms. In the former, the parameter space spans a finite range in σ so that the quantization gives discrete modes. Continuous modes will result if the boundaries are allowed to go to infinity. In the Thirring-model formalism, the parameter space covers the whole (σ, τ) plane. Discrete modes are obtained because the quantization is carried out in the Euclidean (σ, τ) space. Wick's rotation back to the Minkowskian (σ, τ) space takes place only after the quantization. If we impose a boundary at finite σ , discrete modes can be obtained directly in the Minkowskian (σ, τ) space.

B. Meaning of the scale dimension

Light-cone or short-distance singularities occur when an expansion of operator products at almost lightlike distances [i.e., $(x-y)^2 \rightarrow 0$] or at short distances (i.e., $x_\mu \rightarrow y_\mu$) is carried out. They have the following form⁴⁰:

$$A(x)B(y) = \sum_{[\alpha]} C^{[\alpha]}(x-y) O^{[\alpha]}(y),$$

where A , B , and $O^{[\alpha]}$ are local operators and $C^{[\alpha]}$ is a singular c -number function. The index $[\alpha]$ characterizes, in general, a whole set of Lorentz as well as internal quantum values; the sum is over a complete set of operators. The degree of singularity of the function $C^{[\alpha]}$ is governed in general by the dynamical dimension of the operators involved. Such dimensionality is, in general,

different from the canonical value normally expected from a formal Lagrangian consideration; hence the name "anomalous dimension." They are dynamically determined quantities. This is supposed to be true even in the case of a scale-invariant theory, and has been verified in two-dimensional field theories. However, there are serious objections to the physical importance of the anomalous dimensions. It is found that scale invariance cannot be maintained in perturbative calculations.⁴¹ This has motivated Jackiw⁴² to suggest that the occurrence of such a dimension is due to reasons that are peculiar to two dimensions, while it is the four-dimensional field theories that have something to do with reality. Mueller and Trueman⁴³ argue, using the massive Thirring model as an example, that anomalous dimension changes under certain gauge transformations and hence is void of physical significance.

According to the duality postulates, we observe that it is the two-dimensional conformally invariant field theories [in order to produce the Hagedorn spectrum (2.21)] that are more closely associated with strong interaction than four-dimensional theories; the introduction of a mass term to a conformally invariant model breaks conformal invariance.

In this work, we see that it is precisely the two-dimensionality and the zero-mass features that permit us to introduce spinor fields with spin other than $\frac{1}{2}$. In fact, if we start with the massive case and then take the zero mass limit, we are constrained to spin $\frac{1}{2}$. This forbids the connection to dual models. The presence of a mass term also provides a scale so that an S matrix in the parameter space can be defined which is void of physical content. To summarize: Known arguments against anomalous dimensions are circumvented by the duality postulates. However, we also observe in this work that the degree of singularity of the function $C^{[\alpha]}$ that is physically relevant is independent of the strength of the coupling constants in the parameter space.

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$$\partial_\mu \Psi = i(2\pi/c)^{1/2} : b \cdot j_+(u) \Psi : ,$$

$$\partial_\nu \Psi = -\frac{1}{2} i : h \cdot j_-(v) \Psi : ,$$
 so that we can investigate this extended Thirring model instead of the one discussed in Sec. V.
 - ³⁷Note that the limits of $\Delta u, \Delta v \rightarrow 0$ are taken before we evaluate the commutator.
 - ³⁸Our formalism does not allow the introduction of a mass m which later approaches zero. See Sec. VIII for a discussion on this point.
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