A Fast Algorithm for Computing Optimal Rectilinear Steiner Trees for Extremal Point Sets¹

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Abstract

We present an efficient algorithm to compute an optimal rectilinear Steiner tree for extremal point sets. A point set is extremal if every point in it lies on the boundary of a rectilinear convex hull of the point set. Our algorithm can be used to construct minimum-length connection for multi-terminal nets in homotopic routing in VLSI layout design. The previous best algorithms run in $O(k^4n)$ time and $O(n^3)$ time, where n is the size of the point set and k is the size of its rectilinear convex hull. Our algorithm runs in $O(k^2n)$ time which represents a significant improvement.

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Routing in VLSI layout design calls for connecting terminals in the same plane with horizontal and vertical wiring. The terminals are points that sit on the boundaries of isothetic modules which are impenetrable rectilinear polygons. So wiring can only be done in the routing region outside the modules. The terminals are divided into groups known as *nets* [13] and the terminals in each net are to be wired together. The connection within a net is a *rectilinear Steiner tree* (RST). A RST of a terminal point set S is a tree with horizontal and vertical edges that spans all *terminal points* in S . Any tree vertex not in S with degree at least three is a *Steiner point*. The total length of wiring is often required to be minimized which corresponds to finding a *minimal* (optimal) rectilinear Steiner tree (MRST). The general MRST problem is known to be NP-complete [7]. In *homotopic routing* [10], the user supplies a sketch of the wiring of the nets which specifies the placement of the modules and the topology of the wiring, but the wires are considered flexible. The problem is to determine the feasibility of the sketch and if feasible, compute the actual positions of the wiring so as to minimize the total wire length. In [9], it is shown that the sketch can be used to automatically partition the routing region outside the modules into rectilinear polygons without holes, such that each polygon contains exactly one net and the connection for each net must lie within its polygon. Thus, homotopic routing requires the solution to a special MRST problem where the terminal points lie on the boundary of a rectilinear polygon. It is commonly assumed that the input specifies the polygon and the terminal points in cyclic order on the boundary of this polygon [4, 12]. This paper describes an efficient algorithm when the terminal points lie on their rectilinear convex hull.

Previous results. Let n be the size of S . Rectangle is the simplest rectilinear polygon. Aho *et al.* [2] are the first to present an $O(n^3)$ -time algorithm for this case. Subsequently, this has been improved to $O(n)$ time by Cohoon *et al* [5] and Agarwal and Shing [1]. Provan [11], Bern [3], and Erickson *et al* [6] propose algorithms for computing optimal graphical Steiner tree [8] for vertices on the outer boundary of a planar graph. Thus, their algorithms work for general rectilinear polygons. Provan’s algorithm runs in $O(n^6)$ time and the algorithms by Bern [3] and Erickson *et al* [6] run in $O(n^5)$ time. When the planar graph is actually a grid graph (which is the case in our problem), Kaufmann *et al* [9] present an algorithm that runs in $O(n^4 \log n)$ time in the worst case. Given the large number of nets that need to be connected, a faster algorithm is desired. For the special case where the terminal points in S lie on the boundary of a rectilinear convex hull of S , Richards and Salowe [12] present an $O(k^4 n)$ -time algorithm, where k is the size of the rectilinear convex hull. When $k = O(1)$, the algorithm runs in linear time. But the running time degrades to $O(n^5)$ when $k = \Omega(n)$. Cheng *et al* [4] improve upon the worst-case and give an $O(n^3)$ -time algorithm.

Our result. We present a dynamic programming $O(k^2 n)$ -time algorithm for the MRST problem when the terminals points in S lie on the boundary of a rectilinear convex hull of S . Our main contribution is the identification of new structures in a MRST (e.g., common substructures, gaps, and divisions) which lead to a faster algorithm. Section 2 gives the basic definitions. Section 3 presents the basic ideas in our algorithm. Section 4 defines the partial solutions needed in dynamic programming. Sections 5–8 provides a more in-depth discussion of the algorithm, proofs of correctness, and time complexity analysis.

2 Definitions

This section provides the definitions of and necessary concepts about rectilinear convex hull, extremal point set, grid graph, and rectilinear Steiner tree. All geometric objects in this paper are rectilinear (i.e., consists of horizontal and vertical line segments). For brevity, we omit ‘rectilinear’ for the rest of this paper. We also assume throughout this paper that S is the input point set.

Convex hull and extremal point set. A *shortest path* between two points consists of an alternating sequence of horizontal and vertical edges such that both the x -coordinate and y -coordinate are monotonically non-decreasing or non-increasing (i.e., the path has the shape of a staircase). A polygon is *convex* if for any two points in it, there is a shortest path between them that lies within the polygon. Given a terminal point set S , a *convex hull* of S is a smallest-area convex polygon containing S . We denote the boundary of the convex hull specified in the input by $RC(S)$. $RC(S)$ consists of four *boundary staircases* B_i , $1 \leq i \leq 4$, which can be obtained by cutting $RC(S)$ at the highest, lowest, leftmost, and rightmost vertices. See Figure 1(a). For simplicity, we assume throughout this paper that $RC(S)$ has no ‘pinched’ vertex. See Figure 1(b) for a convex hull with pinched vertices. This assumption can be removed by an $O(n)$ -time preprocessing [12]. An *inner boundary corner* is a reflex corner on $RC(S)$. An *outer boundary corner* is a convex corner on $RC(S)$. Note that an outer boundary corner vertex must be a terminal point. A *boundary edge* is a maximal straight line segment on $RC(S)$. S is *extremal* if every terminal point in S lies on $RC(S)$.

Figure 1:

Grid graph. A *node* is either a terminal point or the vertex at an inner boundary corner. A *grid line* is a line segment with endpoints on $RC(S)$ such that its interior lies inside $RC(S)$ and at least one endpoint is a node. The two endpoints of a grid line are called *exterior vertices*. Note that a node is an exterior vertex but not necessarily vice versa. A *grid graph* is the union of $RC(S)$ and all possible horizontal and vertical grid lines. The intersection of two grid lines inside $RC(S)$ is a *grid vertex*. Given exterior vertices c and d , $B(c, d)$ is the portion of $RC(S)$ traversed from c to d in the counterclockwise order. Define $[c, d]$ to be the counterclockwise ordered sequence of exterior vertices on $B(c, d)$. Define $S[c, d]$ to be subsequence of $[c, d]$ that contains c, d , and all terminal points in $[c, d]$. See Figure 1(a) for an example: $[c, d] = \{c, v_1, v_2, v_3, v_4, v_5, v_6, d\}$ and $S[c, d] = \{c, v_1, v_5, d\}$. Note that for any exterior vertices c and d , $S[c, d]$ is extremal.

Steiner tree. A Steiner tree vertex is either a terminal point, the intersection of two orthogonal line segments contained in the tree, or the vertex at an inner boundary corner. A tree edge is a line segment in the tree incident to two adjacent vertices. A Steiner point is a vertex that is not a terminal point and has degree at least three. An *interior edge* is a tree edge whose interior lies inside $RC(S)$. An *interior line* is a maximal sequence of adjacent and collinear interior edges. A *complete interior line* is an interior line whose endpoints lie on $RC(S)$. If c and d are endpoints of an interior line, then we denote the interior line by (c, d) or (d, c) . A degree two vertex incident to two orthogonal tree edges is a *corner-vertex*. An *interior corner* is a corner whose corner-vertex lies inside $RC(S)$. The two interior lines incident to an interior corner-vertex are the *legs* of the interior corner. If both legs of an interior corner intersect $RC(S)$, it is called a *complete interior corner*. Three interior edges incident to the same vertex form a T and the vertex is a *T-vertex*. The interior line that contains the two collinear interior edges is the *head* of the T or T-head. The interior line containing the other interior edge is the *body* of the T or T-body. Yang and Wing [14] prove that there is an optimal Steiner tree which is a subgraph of the grid graph. Therefore, we will assume that all Steiner trees considered in this paper are subgraphs of the grid graph. For any exterior vertices c and d , whenever we refer to any ‘interior edge/line/corner’ of a Steiner tree for $S[c, d]$, we always refer to the interior of $RC(S)$ (not $RC(S[c, d])$). Let $\tau(S[c, d])$ be the set of all Steiner trees for $S[c, d]$ which are subgraphs of the grid graph.

3 Basic ideas

For each inner boundary corner of $RC(S)$, we shall assign a blue color to the two grid lines incident to the corner vertex and a constant number of grid lines nearby. Thus, we shall get $O(k)$ blue grid lines, where k is the size of $RC(S)$. A line/edge is blue if it lies on a blue grid line. A complete interior corner is blue if its two legs are blue. An exterior vertex is blue if it is the endpoint of a blue grid line or it is the vertex at an outer boundary corner. Our coloring procedure is an extension of that in [12]. In [12], for each inner boundary corner, the corner vertex and a constant number of terminal points are colored blue. Our procedure looks at the same set of terminal points for each inner boundary corner, but we color the grid lines incident to the corner vertex and these terminal points blue. Thus, our set of blue exterior vertices is a superset of that in [12]. This does not affect the properties of MRST proved in [12]. Due to lack of space, we omit the details of the color assignment in [12] as our algorithm does not depend on it. We want to point out that grid lines are colored blue only once by inspecting $RC(S)$. We may refer to the colors of vertices/edges/lines of a Steiner tree in $\tau(S[c, d])$ for some exterior vertices c and d . Nevertheless, the coloring is always

done based on $RC(S)$ (not $RC(S[c, d])$). The situation is analogous to the meaning of interior edge/line/corner in a Steiner tree in $\tau(S[c, d])$. Since all vertices at the corners of $RC(S)$ are blue, we have the following useful fact:

Fact: Let c and d be exterior vertices such that $[c, d]$ contains a blue exterior vertex. Let v be the blue exterior vertex in the sequence $[c, d]$ closest to c . Then c and v are contained in the same boundary edge.

Figure 2:

It is proved in [12] that by sliding interior lines and flipping interior corners (Figure 2), there exists a subset of optimal Steiner trees in $\tau(S)$ that satisfy what we call the *blue topological properties* :

Blue topological properties:

- B1** If v_1 and v_2 are two adjacent Steiner points in the interior of an interior line ℓ , then both v_1 and v_2 have degree three. Moreover, the two interior lines properly incident⁴ to ℓ at v_1 and v_2 must be on opposite sides of ℓ . That is, the interior lines properly incident to ℓ *alternate* between the two sides of ℓ .
- B2** The body of a T is an interior edge (i.e., it contains no Steiner point in its interior) and it is incident to a node on the boundary.
- B3** Every interior corner must be a complete interior corner. Moreover, at most one of the two legs can have more than one interior edge properly incident to it. Each such properly incident edge is a T-body and is incident to a node on the boundary by B2. Among the interior edges properly incident to a leg ℓ , the interior edge closest to the corner-vertex must lie on the side of ℓ opposite to the side of ℓ containing the other leg.
- B4** No complete interior corner can be flipped to increase overlapping with the boundary, i.e., $RC(S)$.
- B5** All legs of complete interior corners are blue. Among all interior edges properly incident to a leg, the one closest to the corner-vertex is blue.
- B6** All complete interior lines whose interior contain Steiner point(s) are blue.

The blue topological properties imply that every interior line is either a complete interior line, a leg of a complete interior corner, or the body of a T. We define three *groups* of Steiner trees for S that satisfy the blue topological properties. Group 1 contains no interior edge. Group 2 contains some complete interior corners. Group 3 contains some complete interior line but no complete interior corner. Thus, it suffices to compute a Steiner tree G_j with total length no greater than the minimal tree in Group j , $1 \leq j \leq 3$. (Computing the minimal tree in Group 3 directly is difficult as it is hard (and unnecessary) to guarantee that there is no complete interior corner.) The minimal tree among G_1, G_2, G_3 is the desired solution.

Figure 3:

The minimal Group 1 tree equals $RC(S)$ minus the longest portion of $RC(S)$ whose interior does not contain any terminal point. This can be determined easily in $O(n)$ time. Group 1 will be dropped from discussion henceforth. Figure 3(a) shows an example of a Group 2 tree and Figure 3(b) shows an example of a Group 3 tree.

⁴An interior line or edge is *properly incident* to another line ℓ if the interior line or edge is incident to the interior of ℓ .

We first analyze the structure of a Group 2 tree which will illustrate most features of our algorithm. In Figure 3(a), let the two legs of the complete interior corner be incident to exterior vertices b_r and b_t on $RC(S)$. Let c be the corner-vertex. If we duplicate the two legs, then we obtain two subtrees T and T' . T is a minimal-length connection of (b_r, c) , (b_t, c) , and $S[b_r, b_t]$. T' is a minimal-length connection of (b_r, c) , (b_t, c) , and $S[b_t, b_r]$. Therefore, if we have computed enough information for us to construct T and T' , then their union form a candidate for G_2 . By trying all possible complete interior corners, G_2 can be identified. We analyze the structures of T and T' below to introduce *common substructures*, *divisions*, *gaps*, and *optimal subtrees*. Section 4 describes their definitions.

Common substructures. Consider T' . If we duplicate the corner-vertex c , then T' is decomposed into two subtrees T'_1 and T'_2 . T'_1 connects (b_t, c) and $S[b_t, a_i]$ and T'_2 connects (b_r, c) and $S[a_{i+1}, b_r]$ for some neighboring points a_i, a_{i+1} in $S[b_t, b_r]$. By B2, T'_1 and T'_2 enjoys the property that every interior line properly incident to the two legs is a single edge that intersects $RC(S)$. We call T'_1 and T'_2 *common substructures*. So T' is the union of two common substructures and the choice of a_i, a_{i+1} must minimize the total length of T' . This can be done by trying all $O(n)$ a_i, a_{i+1} in $S[b_t, b_r]$.

Divisions, gaps and optimal subtrees. Consider T . W.l.o.g, by B3, we assume that there is exactly one interior edge f properly incident to (b_r, c) that has an endpoint b_x on $S[b_r, b_t]$. If we duplicate the vertices b_r, b_x and c , then T is decomposed into three parts T_1, T_2 and T_3 . T_1 is $\{f, (b_r, c)\}$. T_2 spans $S[b_r, b_x]$ and contains two connected components T_{21} and T_{22} , which are optimal Steiner trees for $S[b_r, a_i]$ and $S[a_{i+1}, b_t]$ for some neighboring points a_i, a_{i+1} in $S[b_r, b_x]$. The choice of a_i, a_{i+1} must minimize the total length of T_2 and we call T_2 a *gap*. We call T_{21} and T_{22} *optimal subtrees*. T_3 spans $S[b_x, b_t]$ and contains two connected components T_{31} and T_{32} . T_{31} is an optimal subtree for $S[b_x, a_j]$ and T_{32} connects (b_t, c) and $S[a_{j+1}, b_t]$ for some neighboring points a_j, a_{j+1} in $S[b_r, b_t]$. The choice of a_j, a_{j+1} must minimize the total length of T_3 and we call T_3 a *division*. Note that by B2, in T_3 , every interior line properly incident to (b_t, c) is a single edge that intersects $RC(S)$. So T is the minimal-length union of a gap, a division and $\{f, (b_r, c)\}$ which can be obtained by trying all possible $O(n)$ locations of b_x . Note that if f does not exist, then we can take $b_x = b_r$, the gap disappears, and T becomes the union of (b_r, c) and a division.

Thus, G_2 can be constructed by composing common substructures, gaps, and divisions. From the above discussion, given a blue complete interior corner, it takes $O(n)$ time to construct a candidate for G_2 . There are $O(k^2)$ blue complete interior corners to consider and so G_2 can be constructed in $O(k^2n)$ time. Consider an example of a Group 3 tree in Figure 3(b). If we duplicate the complete interior line (b_r, b_t) , then we obtain two subtrees T and T' . T connects (b_r, b_t) and $S[b_r, b_t]$ and T' connects (b_t, b_r) and $S[b_t, b_r]$. If we duplicate b_t in T and T' , then both T and T' become divisions and they can be put together in $O(1)$ time. However, it is too time-consuming to compute all common substructures, divisions, gaps, and optimal subtrees. Therefore, the construction of G_2 and G_3 may sometimes be done indirectly. It still takes $O(k^2n)$ time to construct G_2 and it will take $O(kn)$ time to construct G_3 . The details are a bit more complicated and are given in appendix A. This summarizes the basic approach to compute G_2 and G_3 .

The remaining issue is how to compute common substructures, divisions, gaps, and optimal subtrees. We first examine the structure of a division closer. Consider the division T_3 again in Figure 3(a) which spans $S[b_x, b_t]$ and contains (b_t, c) . Consider the interior edge properly incident to (b_t, c) and furthest from b_t . Let its endpoint on $RC(S)$ be b_y . In T_{32} , if we duplicate b_y , then we obtain an optimal subtree T_{321} which interconnects $S[a_{j+1}, b_y]$ and a common substructure T_{322} which connects (b_t, c) and $S[b_y, b_t]$. T_{31} and T_{321} form a gap for $S[b_x, b_y]$. Thus, a division can be defined to be the union of a gap and a common substructure. Section 4 presents the definitions of common substructure, gap, division, and optimal subtree. The algorithms to construct them are described in Sections 5– 8.

4 Partial solutions

In the following, let b_r and b_t be two exterior vertices such that $[b_r, b_t]$ lie on three boundary staircases (i.e., $[b_r, b_t] \subseteq B_1 \cup B_2 \cup B_3$, or $B_2 \cup B_3 \cup B_4$, or $B_3 \cup B_4 \cup B_1$, or $B_4 \cup B_1 \cup B_2$.)

Optimal subtrees. Denoted by $Tree(b_r, b_t)$. $Tree(b_r, b_t) \in \tau(S[b_r, b_t])$ and either at least one of b_r and b_t is blue, or (b_r, b_t) is a non-blue grid line.

Gap. Denoted by $Gap(b_r, b_t)$. $Gap(b_r, b_t) = T_1 \cup T_2$ where $T_1 \in \tau(S[b_r, a_i])$ and $T_2 \in \tau(S[a_{i+1}, b_t])$ such that a_i, a_{i+1} are neighboring points in $S[b_r, b_t]$ that minimizes the total length of $Gap(b_r, b_t)$. At least one of b_r and b_t must be blue.

Common substructure. Denoted by $C(b_r, b_t, (b_r, v))$. Given an interior line $\ell = (c, d)$, let $openspan(\ell)$ be the closed half plane that contains c and bounded by an infinite line through d orthogonal to ℓ .⁵ $C(b_r, b_t, (b_r, v))$ is defined when v is a grid vertex or exterior vertex, (b_r, v) is a blue interior line and $[b_r, b_t] \subseteq openspan((b_r, v))$. $C(b_r, b_t, (b_r, v))$ is the minimal Steiner tree that connects (b_r, v) and $S[b_r, b_t]$, and satisfies the following requirement: every interior line incident to v or properly incident to (b_r, v) is a single edge which is orthogonal to (b_r, v) and incident to an exterior vertex in $[b_r, b_t]$. We can similarly define a ‘clockwise’ version $C_c(b_r, b_t, (b_r, v))$ to connect (b_r, v) and $S[b_t, b_r]$ if $[b_t, b_r]$ lies on three boundary staircases and $[b_t, b_r] \subseteq openspan((b_r, v))$. The same requirements hold for v and (b_r, v) , and every interior line incident to v or properly incident to (b_r, v) must be a single edge which is orthogonal to (b_r, v) and incident to an exterior vertex in $[b_t, b_r]$.

Division. Denoted by $Div(b_r, b_t, (b_r, v))$. $Div(b_r, b_t, (b_r, v))$ is defined when b_t is blue, v is a grid vertex or an exterior vertex, (b_r, v) is a blue interior line, and $[b_r, b_t] \subseteq openspan((b_r, v))$ or $b_t \notin openspan((b_r, v))$. $Div(b_r, b_t, (b_r, v))$ equals $C(b_r, b_x, (b_r, v)) \cup Gap(b_x, b_t)$ where $b_x \in [b_r, b_t] \cap openspan((b_r, v)) - \{b_t\}$ gives the minimum total length among all exterior vertices in $[b_r, b_t] \cap openspan((b_r, v)) - \{b_t\}$. We can similarly define a ‘clockwise’ version $Div_c(b_r, b_t, (b_r, v))$ if $[b_t, b_r]$ lies on three boundary staircases. The same requirements hold for b_r, v and b_t . Either $[b_t, b_r] \subseteq openspan((b_r, v))$ or $b_t \notin openspan((b_r, v))$. The formula for $Div_c(b_r, b_t, (b_r, v))$ is the same except that $[b_r, b_t]$ is replaced by $[b_t, b_r]$.

We emphasize again that $[b_r, b_t]$ must lie on three boundary staircases in the above definitions. This is required so that all common substructures, divisions, gaps, and optimal subtrees can be computed in $O(k^2n)$ time. Their computation is based on dynamic programming which has $O(kn)$ iterations. In iteration j , we compute the above structures for all $[b_r, b_t]$ such that $||[b_r, b_t]|| = j$ (and subject to the requirements in the definitions), while these structures have been computed for all $[b_r, b_t]$ such that $||[b_r, b_t]|| < j$ (and subject to the requirements in the definitions). For common substructures $C(b_r, b_t, (b_r, v))$ and divisions $Div(b_r, b_t, (b_r, v))$ (and their clockwise versions), we also need to vary v . For divisions, there are altogether $O(k^2n)$ things to compute as b_r and b_t are blue and there are $O(n)$ possible locations for v . For common structures, there would be too many ($O(kn^2)$) things to compute as b_r is blue and there are $O(n^2)$ possible combinations of b_t and v . So only a subset ($O(kn)$) will be computed explicitly. Other common substructures will be constructed on-line in $O(1)$ time when needed. In each iteration, due to dependency, we must compute gaps first, and then divisions, optimal subtrees, and common substructures in this order.

In dynamic programming, we will build up a ‘bigger’ partial solution (Steiner tree or forest) by taking the union of a constant number of ‘smaller’ partial solutions and a constant number of additional tree edges. This union can be implemented in $O(1)$ time by creating a new record for the ‘bigger’ partial solution which stores pointers to the records for the constituent ‘smaller’ partial solutions and a constant-size list of the additional tree edges. We denote this union by the standard operator $\cup$.

5 Gaps

If both b_r and b_t are blue, $Gap(b_r, b_t)$ can be computed by returning the minimal $Tree(b_r, a_i) \cup Tree(a_{i+1}, b_t)$ for some neighboring points $a_i, a_{i+1} \in S[b_r, b_t]$. This takes $O(n)$ time. For the case where either b_r or b_t is blue and $Gap(b_r, b_t)$ satisfies blue topological properties, we show below that $Gap(b_r, b_t)$ can be constructed in $O(1)$ time. Therefore, the total time to compute all gaps is $O(k^2n)$. The additional requirement of blue topological properties is for simplifying the algorithm. If it is not satisfied, then the gap computed may not have the minimal total length. Nevertheless, recall that gaps are used as building blocks for an optimal Steiner tree that satisfies the blue topological properties. Therefore, incorrect gaps cannot be part of the desired optimal Steiner tree. So we are not concerned about constructing them incorrectly. W.l.o.g., assume that b_r is blue and so b_t is non-blue.

In $Gap(b_r, b_t)$, there are three possible topologies: (1) b_t has degree zero, (2) b_t has degree one and b_t is not incident to any interior line, (3) b_t is incident to an interior line ℓ . We compute in $O(1)$ time three forests F_1, F_2 and F_3 for the three cases and return the minimal one as $Gap(b_r, b_t)$. In case (1), F_1 is $Gap(b_r, a_j) \cup \{b_t\}$, where a_j is the terminal point in $S[b_r, b_t]$ closest to b_t . In case (2), b_t has degree one and $Gap(b_r, b_t)$ contains $B(b_{t-1}, b_t)$. Thus, F_2 equals $Gap(b_r, b_{t-1}) \cup B(b_{t-1}, b_t)$. In case (3), since b_t is non-blue, ℓ is either a complete interior line or the body of a T (by B5, ℓ cannot be a leg of a complete interior corner). We compute in $O(1)$ time two forests F_{31} and F_{32} for these two cases and return the minimal one as F_3 . If ℓ is a complete interior line, then ℓ does not contain any terminal point by B6 (b_t is non-blue). Therefore, F_{31} equals the minimal $Gap(b_r, b_z) \cup Tree(b_z, b_t)$, where $b_z \in [b_r, b_t]$ and (b_z, b_t) is a grid line. (There are at most two such b_z ’s.) Suppose that ℓ is the body of a T. Let b_x be the blue exterior vertex in $[b_r, b_t]$ closest to b_r . Either b_x and b_r belong to the same connected component or they belong to

⁵Note that $openspan((c, d)) \neq openspan((d, c))$.

different components. Recall the fact stated in Section 5 that v_r and v_x are contained in the same boundary edge. Thus, F_{32} equals the minimal forest between $B(b_r, b_x) \cup \text{Gap}(b_x, b_t)$ and $\text{Gap}(b_r, b_x) \cup \text{Tree}(b_x, b_t)$.

6 Divisions

Let $\ell = (b_r, c)$ be a blue complete interior line. Note that it is not necessary that $c \in [b_r, b_t]$. Let $v_0, v_1, v_2 \dots$ be the sequence of grid graph vertices on ℓ at increasing distances from b_r and $v_0 = b_r$. (Note that we do not need to compute the grid graph vertices explicitly. We only need to store the x -coordinates (y -coordinates) of all vertical (horizontal) grid lines.) We show how to compute $\text{Div}(b_r, b_t, \ell_j)$ for all $\ell_j = (b_r, v_j)$ such that $[b_r, b_t] \subseteq \text{openspan}(\ell_j)$ or $b_t \notin \text{openspan}(\ell_j)$ in $O(n)$ time. Thus, the total time to compute all divisions is $O(k^2 n)$. The clockwise versions can be computed similarly. The computation is inductive on j . $\text{Div}(b_r, b_t, \ell_0)$ equals $\text{Gap}(b_r, b_t)$. Consider $j \geq 1$. If $[b_r, b_t] \not\subseteq \text{openspan}(\ell_j)$ and $b_t \in \text{openspan}(\ell_j)$, then we can stop the inductive computation. Otherwise, there are two possible topologies in $\text{Div}(b_r, b_t, \ell_j)$: (1) v_j has degree one, (2) v_j has degree two. We compute in $O(1)$ time two forests F_1 and F_2 for the two cases and return the minimal one as $\text{Div}(b_r, b_t, \ell_j)$. In case (1), by optimality, removing from $\text{Div}(b_r, b_t, \ell_j)$ the part of ℓ_j joining v_{j-1} and v_j should produce $\text{Div}(b_r, b_t, \ell_{j-1})$. Thus, F_1 equals $\text{Div}(b_r, b_t, \ell_{j-1}) \cup \{(v_{j-1}, v_j)\}$. In case (2), v_j is incident to an interior edge which intersects $B(b_r, b_t)$ at an exterior vertex, say b_x . So F_2 equals $C(b_r, b_x, \ell_j) \cup \text{Gap}(b_x, b_t)$. To retrieve b_x , it suffices to retrieve the endpoints of the grid line through v_j perpendicular to ℓ . This can be done by storing the horizontal (vertical) grid lines in an array indexed by their y -coordinates (x -coordinates).

7 Optimal subtrees

We first define four subsets of $\tau(S[b_r, b_t])$ (possibly non-disjoint and empty) when b_r or b_t is blue. W.l.o.g., we assume throughout this section that $[b_r, b_t]$ lies on $B_1 \cup B_2 \cup B_3$. The other cases can be handled similarly. In the following, let $\{b_r, b_{r+1}, \dots, b_{t-1}, b_t\}$ be the sequence $[b_r, b_t]$. Let $\beta(S[b_r, b_t]) \subseteq \tau(S[b_r, b_t])$ be the set of all trees that satisfy the blue topological properties.

Class-1. Let b_x be the blue exterior vertex in $[b_r, b_t]$ closest to b_r . Let b_y be the blue exterior vertex in $[b_r, b_t]$ closest to b_t . If b_r is blue, then the Class-1 trees are $\text{Tree}(b_r, b_{t-1}) \cup B(b_{t-1}, b_t)$ and $B(b_r, b_x) \cup \text{Tree}(b_x, b_t)$. If b_t is blue, then the Class-1 trees are $B(b_r, b_{r+1}) \cup \text{Tree}(b_{r+1}, b_t)$ and $\text{Tree}(b_r, b_y) \cup B(b_y, b_t)$. If both b_r and b_t are blue, then the Class-1 trees are the above four trees.

Class-2. A Steiner tree in $\beta(S[b_r, b_t])$ that contains a blue complete interior line incident to b_r or b_t .

Class-3. A Steiner tree in $\beta(S[b_r, b_t])$. If there exists $b_x \in [b_r, b_t] - \{b_r, b_t\}$ such that (b_r, b_x) or (b_t, b_x) is a grid line, then $\text{Tree}(b_r, b_x) \cup \text{Tree}(b_x, b_t)$ is a Class-3 tree. Note that there are at most two such choices for b_x for any $[b_r, b_t]$.

Class-4. A Steiner tree in $\beta(S[b_r, b_t])$ that contains a blue complete interior corner such that one of its legs is incident to b_r or b_t .

Theorem 7.1 *Let $E(\text{Class-1}) \subseteq \tau(S[b_r, b_t])$ be the set of all trees that can be transformed into a Class-1 tree by sliding edge(s) and/or flipping corner(s). If b_r or b_t is blue and $\text{Tree}(b_r, b_t) \in \beta(S[b_r, b_t]) - E(\text{Class-1})$, then $\text{Tree}(b_r, b_t) \in \bigcup_{j=2}^4 \text{Class-}j$.*

Proof W.l.o.g., assume that b_r is blue (b_t may or may not be blue). Suppose that $\text{Tree}(b_r, b_t) \in \beta(S[b_r, b_t]) - E(\text{Class-1})$. $\text{Tree}(b_r, b_t)$ must contain some interior line ℓ incident to b_t . Otherwise, $\text{Tree}(b_r, b_t)$ equals $\text{Tree}(b_r, b_{t-1}) \cup B(b_{t-1}, b_t)$ which is in Class-1, a contradiction. By blue topological properties, ℓ is a complete interior line, a leg of a complete interior corner, or the body of a T. If ℓ is a complete interior line, then $\text{Tree}(b_r, b_t)$ is in Class-2 or Class-3. If ℓ is a leg of a complete interior corner, then $\text{Tree}(b_r, b_t)$ is in Class-4. If ℓ is the body of an interior T, then ℓ must be horizontal due to the fact that $[b_r, b_t]$ lie on $B_1 \cup B_2 \cup B_3$ and $\text{Tree}(b_r, b_t)$ lies inside $RC(S[b_r, b_t])$. We claim that the upper endpoint of the T-head is an exterior vertex b_p on the boundary as shown in Figure 4(a). By B5 or B6, b_p is blue. Moreover, $b_p = b_r$. (Otherwise, $\text{Tree}(b_r, b_t)$ equals $B(b_r, b_p) \cup \text{Tree}(b_p, b_t)$ which is in Class-1, a contradiction). This implies that $\text{Tree}(b_r, b_t)$ is in Class-2 or Class-4. If the upper endpoint of the T-head is not on the boundary, then it is an interior corner-vertex. There are two possible configurations as shown in Figures 4(b) and (c), depending on whether there is any interior edge properly incident to the horizontal leg. We derive below a contradiction in both cases. By B5, b_q in Figures 4(b) and (c) is blue. Also, $b_q \neq b_r$ as $\text{Tree}(b_r, b_t)$ would lie outside $RC(S[b_r, b_t])$ otherwise. We flip the complete interior corner (in Figure 4(c), b_q must lie on B_3 ; otherwise, this flipping would increase overlapping with the boundary which violates B4). After the flipping, we obtain the tree $B(b_r, b_q) \cup \text{Tree}(b_q, b_t)$ which is in Class-1. This contradicts that $\text{Tree}(b_r, b_t) \notin E(\text{Class-1})$. $\square$

Figure 4:

Theorem 7.1 means that if b_r or b_t is blue, then it suffices to construct a Steiner tree CT_j whose total length is not greater than that of the minimal Steiner tree in Class- j , for $1 \leq j \leq 4$. The minimal tree among CT_j , $1 \leq j \leq 4$, is returned as $Tree(b_r, b_t)$. We do not construct the minimal Steiner tree in Class- j , $1 \leq j \leq 4$, directly because we will slide edge(s) to transform them so that we can use ‘smaller’ partial solutions. If $\beta(S[b_r, b_t]) - E(\text{Class-1})$ is empty, then the $Tree(b_r, b_t)$ constructed may not have the minimal total length. Nevertheless, recall that optimal subtrees are used as building blocks for an optimal Steiner tree that satisfies the blue topological properties. Therefore, incorrect optimal subtrees cannot be part of the desired optimal Steiner tree. So we are not concerned about constructing them incorrectly. By similar argument, we also make the following remark:

Remark. When (b_r, b_t) is a non-blue grid line, it suffices to construct a $Tree(b_r, b_t)$ that also belongs to $\beta(S[b_r, b_t])$. Sections 7.1–7.4 describe how to compute CT_j , $1 \leq j \leq 4$, in $O(k)$ time. Section 7.5 describes how to compute $Tree(b_r, b_t)$ in $O(1)$ time when (b_r, b_t) is a non-blue grid line. Thus, the total time to compute all optimal subtrees is $O(k^2n)$.

7.1 Class-1

We compute at most four candidate trees as described in the definition of Class-1 and pick the minimal one as CT_1 . To do this in $O(1)$ time, at each exterior vertex, we need to keep pointers to the neighboring exterior vertices and neighboring blue exterior vertices. This can be done in preprocessing by an $O(n)$ -time scan of all exterior vertices in cyclic order around $RC(S)$.

7.2 Class-2

For each blue grid line (b_r, b_x) or (b_t, b_x) , where $b_x \in [b_r, b_t]$, we compute a candidate tree in $O(1)$ time. There are at most two such blue grid lines for any $[b_r, b_t]$. The minimal candidate tree is returned as CT_2 . Suppose that the minimal Class-2 tree contains (b_r, b_x) . (The case for (b_t, b_x) is symmetric.) If $b_t \in \text{openspan}((b_x, b_r))$, the corresponding candidate tree equals $\text{Div}(b_r, b_x, (b_r, b_x)) \cup C(b_x, b_t, (b_x, b_r))$. The union takes $O(1)$ time. This processing demands that for any exterior vertex c , we keep pointer(s) to the other endpoint(s) of the vertical/horizontal grid line(s) incident to c . Since there are $O(n)$ grid lines, this can be done in $O(n)$ time in preprocessing. When $b_t \notin \text{openspan}((b_x, b_r))$, b_r must be on B_3 , (b_x, b_r) must be vertical, and b_t must be on B_1 and above b_r . Thus, $RC(S[b_r, b_t]) = RC(S[b_r, b_{t-1}]) \cup B(b_{t-1}, b_t)$ and so $Tree(b_r, b_t)$ equals $Tree(b_r, b_{t-1}) \cup B(b_{t-1}, b_t)$ which is in Class-1. Class-1 has been handled before and so we can ignore the case $b_t \notin \text{openspan}((b_x, b_r))$.

7.3 Class-3

CT_3 is the minimal $Tree(b_r, b_x) \cup Tree(b_x, b_t)$, where $b_x \in [b_r, b_t]$ and (b_r, b_x) or (b_t, b_x) is a grid line (not necessarily blue). There are at most two such grid lines for any $[b_r, b_t]$. Note that if b_r is blue, then b_x is also blue and so both $Tree(b_r, b_x)$ and $Tree(b_x, b_t)$ are available. If b_r is non-blue, then (b_r, b_x) is a non-blue grid line and b_t must be blue by definition. So $Tree(b_r, b_x)$ and $Tree(b_x, b_t)$ are also available. (Recall that Section 7.5 describes the computation of $Tree(b_r, b_t)$ when (b_r, b_t) is a non-blue grid line.)

7.4 Class-4

There are four possible configurations depending on the orientation of the blue complete interior corner. See Figure 5. We compute four candidate trees in $O(k)$ time which are in the four configurations shown. The minimal tree is returned as CT_4 . Configurations A and B are symmetric and configurations C and D are symmetric. Therefore, we only discuss how to compute the candidate trees for configurations A and C.

Figure 5:

Figure 6:

7.4.1 Configuration A

Let c be the corner-vertex and let the vertical leg be incident to $b_s \in [b_r, b_t]$. Let (b_t, c, b_s) denote the blue complete interior corner. So b_s and b_t are blue but the position of b_s is unknown. Recall that $E(\text{Class-1}) \subseteq \tau(S[b_r, b_t])$ is the set of all trees that can be transformed into a Class-1 tree by sliding edge(s) and/or flipping corner(s).

Lemma 7.1 *Let T be a tree of configuration A and $T \notin E(\text{Class-1})$. Then b_r has degree one in T and b_r is the endpoint of an interior edge properly incident to (b_t, c) .*

Proof In T , b_r must be the endpoint of an interior line ℓ . Otherwise, b_r is only incident to one tree edge that contains $B(b_r, b_{r+1})$ and so T equals $B(b_r, b_{r+1}) \cup \text{Tree}(b_{r+1}, b_t)$ which is in Class-1, a contradiction. Since T must lie inside $RC(S[b_r, b_t])$, ℓ must be vertical. By blue topological properties, ℓ is either a complete interior line, a leg of a complete interior corner, or the body of a T. ℓ is not a complete interior line because c would then be the endpoint of a T-body which contradicts B2. ℓ is not a leg of a complete interior corner because that complete interior corner can then be flipped to overlap with the boundary. This contradicts B4. Thus, ℓ must be a T-body properly incident to (b_t, c) and by B2, ℓ is an edge. If b_r has degree more than one, then b_r is incident to a tree edge that contains $B(b_r, b_{r+1})$. But then we can slide ℓ to become incident to b_{r+1} and obtain a Class-1 tree, a contradiction. $\square$

Since we know how to handle Class-1, we will make the assumptions about b_r as stated in Lemma 7.1. By B3, either (1) there is exactly one interior edge properly incident to (b_t, c) (with b_r as one endpoint), or (2) there is at most one interior edge properly incident to (b_s, c) . Refer to Figures 6(a) and 6(b) for the two cases. We compute in $O(k)$ time two trees T_1 and T_2 for cases (1) and (2) and return the minimal one as the desired tree of configuration A.

Case (1): Figure 6(a). By B5, b_r is blue. Thus, the tree equals $\text{Div}_c(b_s, b_r, (b_s, c)) \cup \text{Div}(b_s, b_t, (b_s, c)) \cup \{(b_r, d), (b_t, c)\}$, which can be constructed in $O(1)$ time if b_s is known. We try all $O(k)$ positions for b_s and return the tree with minimal total length as T_1 .

Case (2): Figure 6(b). Let b_x be the left endpoint of the interior edge properly incident to (b_s, c) . By B5, b_x is blue. If no interior edge is properly incident to (b_s, c) , we take b_x to be b_s . By B1, interior edges properly incident to (b_t, c) are alternating. We can assume that there is one less interior edge properly incident to the lower side of (b_t, c) than its upper side. (Otherwise, we can slide the horizontal leg downward without increasing total length until the horizontal leg becomes incident to the blue exterior vertex in $[b_r, b_t]$ closest to b_t . This converts the Steiner tree into a Class-1 tree which has been handled before.) By our assumption, we can convert Figure 6(b) to Figure 6(c). Let a_p be the terminal point in $S[b_r, b_t]$ closest to b_r . The tree in Figure 6(c) equals $C_c(b_x, a_p, (b_x, w)) \cup \text{Div}(b_x, b_t, (b_x, w)) \cup \{(b_r, w), (b_t, v)\}$, which can be constructed in $O(1)$ time if b_x is known. We try all $O(k)$ positions for b_x and return the tree with minimal total length as T_2 .

7.4.2 Configuration C

Let c be the corner-vertex and let the horizontal leg be incident to $b_s \in [b_r, b_t]$. Let (b_r, c, b_s) denote the blue complete interior corner. So b_r and b_s are blue but the position of b_s is unknown. There are two cases: (1) $b_s = b_t$ (if b_t is blue),

Figure 7:

(2) $b_s \neq b_t$. We compute in $O(k)$ time two trees T_1 and T_2 for the two cases and return the minimal one as the desired tree of configuration C.

Consider T_1 . By B3, one leg of (b_r, c, b_t) has at most one interior edge properly incident to it. There are four different cases depending on (b_r, c) or (b_t, c) has zero or one interior edge properly incident to it. We compute a candidate tree for each case and then return the minimal tree as T_1 . For the case where no interior edge is properly incident to (b_t, c) , the candidate tree equals $Div(b_r, b_t, (b_r, c)) \cup \{(b_t, c)\}$. Consider the case where exactly one interior edge f is properly incident to (b_t, c) . Let b_x be the endpoint of f on the boundary which is blue by B5. If b_x is known, then the candidate tree equals $Div(b_r, b_x, (b_r, c)) \cup Gap(b_x, b_t) \cup \{(b_t, c), f\}$. We try all $O(k)$ positions for b_x in $[b_r, b_t]$ to find the position that minimizes the total length. This takes $O(k)$ time. The two cases where at most one interior edge is properly incident to (b_r, c) is handled similarly.

Consider T_2 . First, by a proof similar to that of Lemma 7.1, we can assume that b_t is the endpoint of an interior edge properly incident to (b_r, c) . By B1, interior edges properly incident to (b_r, c) are alternating. By B2, B3 and B5, there is at most one interior edge properly incident to (b_t, c) and this interior edge (if exists) is incident to a blue node b_x . If no interior edge is properly incident to (b_t, c) , then we take b_x to be b_s . There are two cases: (1) same number of interior edges are properly incident to both sides of (b_r, c) (Figure 7(a)), (2) there is one more interior edge properly incident to the left side of (b_r, c) than the right side (Figure 7(b)). We compute in $O(k)$ time two trees T_{21} and T_{22} and return the minimal one as T_2 . In case (1), we convert Figure 7(a) to Figure 7(c). T_{21} , which is the tree in Figure 7(c), equals $Div_c(b_x, b_r, (b_x, w)) \cup C(b_x, b_t, (b_x, b_t)) \cup \{(b_r, v), (v, w)\}$, which can be constructed in $O(1)$ time if the position of b_x is known. We try all $O(k)$ positions for b_x in $[b_r, b_t]$ to find the position that minimizes the total length. In case (2), there is one interior edge, say f , properly incident to (b_r, v) . Let b_z be the exterior vertex in $[b_r, b_t]$ such that (b_z, b_t) is a horizontal grid line. Let b_p be last node in the sequence $[b_r, b_z]$ (recall that a node is a terminal point or an inner boundary corner vertex). Let b_q be the lower endpoint of the vertical boundary edge that contains b_z . Let a_p (a_q) be the terminal point closest to b_p (b_q) in $S[b_p, b_x]$ ($S[b_q, b_x]$). Let g be the interior edge immediately below f and properly incident to (v, c) . The following lemma can be proved (the proof is given in appendix B).

Lemma 7.2 *Let T be a tree of configuration C such that T cannot be converted to a Class-1 tree or a tree of configuration A by sliding edges and flipping corners. If the total length of T equals that of $Tree(b_r, b_t)$, then one of the following holds:*

1. f can be slid downward until it contains b_p , and after the sliding, b_p (a_p) is connected by a tree path in T to (b_r, v) ((v, c)) that uses f (g),
2. b_q is a terminal point, f can be slid downward until it contains b_z , and b_q (a_q) is connected by a tree path in T to (b_r, v) ((v, c)) that uses f (g).

We convert Figure 7(b) to Figure 7(d) by sliding edge(s). T_{21} to be computed is the tree in Figure 7(d). We assume that the total of the tree in Figure 7(d) equals to that of $Tree(b_r, b_t)$. Otherwise, there is no point in computing T_{21} . By definition and Lemma 7.2, a_p and a_q are below v and so $a_p, a_q \in openspan((b_x, w))$. Lemma 7.2 implies that $S[a_p, b_x]$ (or $S[a_q, b_x]$) is connected together with (b_x, w) to form a common substructure $C_c(b_x, a_p, (b_x, w))$ ($C_c(b_x, a_q, (b_x, w))$). Given b_t , we can access b_z and hence b_p in $O(1)$ time. By an $O(n)$ -time preprocessing, we can keep pointers at each exterior vertex c to the two endpoints of the boundary edge containing c . Thus, b_q can also be accessed in $O(1)$ time. We compute in $O(k)$ time two trees T_{221} and T_{222} for the two cases stated in Lemma 7.2 and return the minimal one as T_{22} . In the first case, we slide f downward until it is incident to b_p . T_{221} , which is the resulting tree, equals $Gap(b_r, b_p) \cup C_c(b_x, a_q, (b_x, w)) \cup C(b_x, b_t, (b_x, w)) \cup \{f, (b_r, v), (v, w)\}$ provided that the position of b_x is known. In the second case, since b_z and b_q are contained by the same boundary edge, the tree path connecting b_q to b_z lies on $B(b_z, b_q)$. We slide f downward until it is incident to b_z . T_{222} , which is the resulting tree, equals $Gap(b_r, b_z) \cup B(b_z, b_q) \cup C_c(b_x, a_q, (b_x, w)) \cup C(b_x, b_t, (b_x, w)) \cup \{f, (b_r, v), (v, w)\}$ provided that the position of b_x is known. We try all $O(k)$ positions for b_x to minimize the total length of T_{221} and T_{222} .

Recall the remark in the beginning of Section 7 that $Tree(b_r, b_t)$ can be assumed to satisfy the blue topological properties. Since (b_r, b_t) is a non-blue grid line, b_r and b_t lie in the interior of two opposite boundary edges. This implies that (b_{r+1}, b_{t-1}) is also a (possibly blue) grid line and $Tree(b_{r+1}, b_{t-1})$ is available. In $Tree(b_r, b_t)$, there are two possible topologies: (1) no interior edge is incident to b_r or b_t , (2) an interior line ℓ is incident to b_r or b_t . We compute in $O(1)$ time two trees T_1 and T_2 for the two cases and return the minimal one as $Tree(b_r, b_t)$. In case (1), T_1 clearly equals $B(b_r, b_{r+1}) \cup Tree(b_{r+1}, b_{t-1}) \cup B(b_{t-1}, b_t)$. In case (2), ℓ cannot be a T-body; otherwise, the T-head must lie partially outside $RC(S[b_r, b_t])$. By B5, since ℓ is non-blue, ℓ cannot be a leg of a complete interior corner. If ℓ is a (non-blue) complete interior line, then $\ell = (b_r, b_t)$ and ℓ does not contain any Steiner point by B6. Let a_q be the terminal point in $S[b_r, b_t]$ closest to b_t . Let (b_s, a_q) be a grid line parallel to ℓ . Let b_p be the blue exterior vertex in $[b_r, b_t]$ closest to b_r . b_r must be connected by a tree path to b_p or b_s , whichever is closer to b_r in $[b_r, b_t]$, such that the tree path lies on the boundary edge containing b_r . Thus, if b_p (b_s) is closer to b_r , then T_2 equals $B(b_r, b_p) \cup Tree(b_p, a_q) \cup \{\ell\}$ ($B(b_r, b_s) \cup Tree(b_s, a_q) \cup \{\ell\}$).

8 Common substructures

Let $\ell = (b_r, v)$ be a blue complete interior line such that $[b_r, b_t] \subseteq openspan(\ell)$. There are at most two such complete interior lines incident to b_r and we compute $C(b_r, b_t, \ell)$ in $O(1)$ time for each such ℓ . For any grid vertex w on ℓ such that $[b_r, b_t] \subseteq openspan((b_r, w))$, whenever $C(b_r, b_t, (b_r, w))$ is needed, it can be constructed on-line in $O(1)$ time as $(C(b_r, b_t, \ell) - \{\ell\}) \cup \{(b_r, w)\}$. The correctness follows from the fact all interior edges incident to ℓ in $C(b_r, b_t, \ell)$ are also incident to (b_r, w) . We describe below how to compute $C(b_r, b_t, \ell)$ in $O(1)$ time. The clockwise versions can be handled similarly. Since there are $O(kn)$ such common substructures, the total processing time is thus $O(kn)$. Let $[b_r, b_t]$ be $\{b_r, b_{r+1}, \dots, b_{t-1}, b_t\}$. W.l.o.g., we assume that ℓ is vertical and b_r is the upper endpoint of ℓ . See Figure 8. There

Figure 8:

are two possible topologies in $C(b_r, b_t, \ell)$: (1) no interior edge is incident to b_t and ℓ , (2) an interior edge f is incident to b_t and ℓ . We compute in $O(1)$ time two trees T_1 and T_2 for the two cases and return the minimal one as $C(b_r, b_t, \ell)$.

Case (1): There are two possible topologies: (1.1) no interior edge is properly incident to ℓ , (1.2) some interior edge is properly incident to ℓ . We compute two trees T_{11} and T_{12} for the two cases and return the minimal one as T_1 . In case (1.1), T_{11} equals $Tree(b_r, b_t) \cup \{\ell\}$. In case (1.2), traverse a tree path P from b_t to b_r until we reach an exterior vertex. We flip all left turns on P and so P is either $B(b_{t-1}, b_t)$ or a complete interior line (b_x, b_t) . See Figures 8(a) and (b). (P cannot contain a right bend; otherwise, there cannot be any interior edge properly incident to ℓ as P has to terminate at an exterior vertex.) Thus, T_{12} equals the minimal tree between $C(b_r, b_{t-1}, \ell) \cup B(b_{t-1}, b_t)$ and $C(b_r, b_x, \ell) \cup Tree(b_x, b_t)$. (There is at most one $b_x \in [b_r, b_t]$ such that (b_x, b_t) is a grid line.)

Case (2): There are three possible topologies: (2.1) b_t has degree one, (2.2) b_t has degree more than one and b_t is not incident to a tree edge perpendicular to f , (2.3) b_t is incident to a tree edge g perpendicular to f . We compute two trees T_{21} and T_{22} for the first two cases and return the minimal one as T_2 . We argue that case (2.3) has been handled before. In case (2.1) (Figure 8(c)), T_{21} equals $C(b_r, a_i, \ell) \cup \{f\}$, where a_i is the terminal point in $S[b_r, b_t]$ closest to b_t . In case (2.2), by convexity, the extra tree edge incident to b_t must contain $B(b_{t-1}, b_t)$ and it is aligned with f . See Figure 8(d). Thus, T_{22} equals the tree T_2 constructed when computing $C(b_r, b_{t-1}, \ell)$. In case (2.3), traverse a tree path P from b_t starting along g and making a right turn whenever possible, until an exterior vertex is reached. Similar to case (1.2) above, after flipping all left turns on P , either P is $B(b_{t-1}, b_t)$ or P is a complete interior line or P contains a right bend as shown in Figure 8(e). In all three cases, f can be slid upward and this reduces case (2.3) to some case considered in case (1).

We present $O(k^2n)$ -time algorithms for computing new structures (common substructures, division, gaps, and optimal subtrees) of a MRST for an extremal point set. Combined with the proof in Appendix A, we obtain an $O(k^2n)$ -time algorithm to compute an optimal Steiner tree for an extremal point set. This strictly improves upon previous results in [12, 4]. When the terminal points are on the boundary of a rectilinear polygon, it is interesting to see if similar techniques can be used to obtain a fast algorithm.

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We first repeat the definitions of Group- j and G_j , $1 \leq j \leq 3$, in Section 3. We define three groups of Steiner trees for S that satisfy the blue topological properties. Group 1 contains no interior edge. Group 2 contains some complete interior corners. Group 3 contains some complete interior line but no complete interior corner. For $1 \leq j \leq 3$, the tree G_j to be computed is a Steiner tree with total length no greater than the minimal tree in Group- j .

We shall show that G_j , $1 \leq j \leq 3$, can be computed in $O(k^2n)$ time. Then the minimal tree among G_1 , G_2 , and G_3 is returned as the desired optimal Steiner tree for the input extremal point set. To this end, we first compute the structures defined in Section 4: common substructures $C(b_r, b_t, (b_r, v))$ (and the clockwise version), divisions $Div(b_r, b_t, (b_r, v))$ (and the clockwise version), gaps $Gap(b_r, b_t)$, and optimal subtrees $Tree(b_r, b_t)$. The readers are referred to Section 4 for their definitions. We just point out here that $[b_r, b_t]$ are required to lie on three boundary staircases⁶. We call this collection of structures *the complete set of partial solutions* and we assume that they have been computed using the algorithms in Section 5–8.

We also need to compute divisions for some more sequences $[b_r, b_t]$. Recall that for $[b_r, b_t]$ lying on three boundary staircases, $Div(b_r, b_t, (b_r, v))$ equals the minimal $C(b_r, b_x, (b_r, v)) \cup Gap(b_x, b_t)$ for some $b_x \in openspan((b_r, v)) \cap [b_r, b_t] - \{b_t\}$. Define $span((b_r, v))$ to be the region between two infinite lines through b_r and v and perpendicular to (b_r, v) . Now, we also compute $Div(b_r, b_t, (b_r, v))$ for $[b_r, b_t]$ such that that: (1) $[b_r, b_t]$ contains exterior vertices from all four boundary staircases, and (2) b_r and b_t do not lie on the same boundary staircase. In such cases, $Div(b_r, b_t, (b_r, v))$ is defined to be the minimal $C(b_r, b_x, (b_r, v)) \cup Gap(b_x, b_t)$ for some $b_x \in span((b_r, v)) \cap [b_r, b_t]$. Similarly, we can define a clockwise version $Div_c(b_r, b_t, (b_r, v))$ to be the minimal $C_c(b_r, b_x, (b_r, v)) \cup Gap(b_t, b_x)$ for some $b_x \in span((b_r, v)) \cap [b_t, b_x]$. We call such an additional division an *extra-division* and we call the collection of all extra-divisions *the complete set of extra-divisions*. Since b_r and b_t are required not to lie on the same boundary staircase, for all $b_x \in span((b_r, v))$, $C(b_r, b_x, (b_r, v))$ and $Gap(b_x, b_t)$ are available as $[b_r, b_x]$ and $[b_x, b_t]$ must lie on three boundary staircases. Thus, extra-divisions can be computed by composing gaps and common substructures which are available. Moreover, since the formula for extra-divisions are very similar to that for divisions, extra-divisions for all appropriate $[b_r, b_t]$ can be computed in $O(k^2n)$ time by a method similar to the one described in Section 6 for divisions.

Theorem A.1 *Given an extremal point set S of size n , the optimal Steiner tree for S can be computed in $O(k^2n)$ time if the complete set of partial solutions and the complete set of extra-divisions are available.*

Proof G_1 is $RC(S)$ minus the longest portion of the boundary between two adjacent terminal points on the boundary. This can be found easily in $O(n)$ time. It suffices to compute G_2 and G_3 such that they satisfy the blue topological properties in Section 3.

Figure 9: G_2 and G_3 .

The minimal Group-2 tree contains a complete interior corner whose legs are incident to two blue exterior vertices b_r and b_t . Let c be the corner-vertex. By B3, there are $O(k^2)$ possible choices for b_r and b_t . For each pair of b_r and b_t , there are three possible cases as listed below. We describe below how to compute in $O(n)$ time a tree for each case. The minimal tree among all the $O(k^2)$ cases is the desired G_2 .

Case (1.1): b_r and b_t lie on opposite boundary staircases (i.e., B_1 and B_3 as in Figure 9(a) or B_2 and B_4). The subtree that connects $S[b_t, b_r]$ and the complete interior corner equals $C(b_t, a_i, (b_t, c)) \cup C_c(b_r, a_{i+1}, (b_r, c))$ for some neighboring points $a_i, a_{i+1} \in S[b_t, b_r]$. The choice of $a_i, a_{i+1} \in S[b_t, b_r]$ should minimize the total length and this can be found in $O(n)$ time. It remains to describe how to compute the subtree connecting $S[b_r, b_t]$ and the complete interior corner. Denote this subtree by T . By B3, there are four possible cases depending on (b_r, c) or (b_t, c) has zero or one interior edge properly incident to it. We compute four candidate trees for the four cases and return the

⁶ $B_1 \cup B_2 \cup B_3$, or $B_2 \cup B_3 \cup B_4$, or $B_3 \cup B_4 \cup B_1$, or $B_4 \cup B_1 \cup B_2$.

minimal one as T . For the case where (b_r, c) has no interior edge properly incident to it, the candidate tree equals $Div_c(b_t, b_r, (b_t, c)) \cup \{(b_r, c)\}$. Suppose that (b_r, c) has one interior edge f properly incident to it. By B2 and B5, f is incident to a blue node b_x in $[b_r, b_t]$. Thus, the candidate tree equals $Gap(b_r, b_x) \cup Div_c(b_t, b_x, (b_t, c)) \cup \{f, (b_r, c)\}$ if the position of b_x is known. We try all $O(k)$ choices for b_x to find the position that minimizes the total length. The candidate trees for the two cases where (b_t, c) has at most one interior edge properly incident to it can be handled similarly.

Case (1.2): b_r and b_t lie on the same boundary staircase. W.l.o.g., assume that b_r and b_t lie on B_4 . See Figure 9(b). As in case (1), the subtree that connects $S[b_t, b_r]$ and the complete interior corner equals the minimal $C(b_t, a_i, (b_t, c)) \cup C_c(b_r, a_{i+1}, (b_r, c))$ for some neighboring points $a_i, a_{i+1} \in S[b_t, b_r]$. By B3 and B4, there must be exactly one interior edge properly incident to (b_r, c) or (b_t, c) so that we cannot flip the complete interior corner to overlap with B_4 . By B2 and B5, such an interior edge is incident to a blue node in $[b_r, b_t]$. Suppose that such an interior edge is properly incident to (b_r, c) and a blue node b_x . Then the subtree that connects $S[b_r, b_t]$ and the complete interior corner equals $Gap(b_r, b_x) \cup Div_c(b_t, b_x, (b_t, c)) \cup \{(b_r, c)\}$. (Note that $[b_x, b_t]$ may not lie on three boundary staircases and in such cases, $Div_c(b_t, b_x, (b_t, c))$ is an extra-division.) Thus, it suffices to try all $O(k)$ positions for b_x to find the position that minimizes the total length.

Case (1.3): b_r and b_t lie on consecutive boundary staircases. For example, b_r lies on B_3 and b_t lies on B_4 . This case can be handled by a method similar to the one in case (2).

The minimal Group-3 tree contains a complete interior line. For each grid line (b_r, b_t) , we compute a minimal tree for the case that (b_r, b_t) is contained in the minimal Group-3 tree. There are $O(n)$ grid lines and the minimal tree among all the $O(n)$ cases is the desired G_3 . For each blue (resp. non-blue) grid line, we show how to compute a minimal tree in $O(n)$ time (resp. $O(1)$ time). Thus, the total processing time is $O(kn)$. When (b_r, b_t) is blue, there are three cases as listed below:

Case (2.1): b_r and b_t lie on opposite boundary staircases. The minimal tree is partitioned along (b_r, b_t) into two divisions: $Div(b_r, b_t, (b_r, b_t)) \cup Div_c(b_r, b_t, (b_r, b_t))$. See Figure 9(e). This takes $O(1)$ time to construct.

Case (2.2): b_r and b_t lie on two consecutive boundary staircases and there exists an interior edge that is properly incident to (b_r, b_t) and intersects a boundary staircase different from the two containing b_r and b_t . W.l.o.g., assume that the configuration looks like Figure 9(c). Among all the interior edges that are properly incident to (b_r, b_t) and intersect a boundary staircase different from the two containing b_r and b_t , let the farthest one from b_r be incident to an exterior vertex b_x in $[b_t, b_r]$. If the position of b_x is known, then the tree in Figure 9(c) equals $Div(b_r, b_t, (b_r, b_t)) \cup C_c(b_r, b_x, (b_r, b_t)) \cup Gap(b_t, b_x)$. We try all $O(n)$ positions for b_x to find the one that minimizes the total length.

Case (2.3): b_r and b_t lie on two consecutive boundary and there is no interior edge that is properly incident to (b_r, b_t) and intersects a boundary staircase different from the two containing b_r and b_t . W.l.o.g., assume that the configuration looks like Figure 9(d). By rotational symmetry, we can assume that there is a longest complete interior line which is vertical. Thus, we can further assume that (b_r, b_t) is a longest complete interior line and it is the rightmost one among all longest vertical complete interior lines. The reason is that such an assumption will definitely satisfy for some choice of (b_r, b_t) as we are exhausting all possible choices. Imagine that a carpet is unrolled starting from (b_r, b_t) to the right until the carpet touches the boundary. The carpet covers a rectangular area and we denote it by R in Figure 9(d). We claim that the interior of R is empty. Assume to the contrary that the interior of R is non-empty. By definition of Group-3, any interior line intersecting R cannot be a leg of an interior corner. Also, we can assume that any interior line intersecting R is not the body of a T; otherwise the T will produce the configuration in Figure 9(c) which have already been handled. Thus, any interior line intersecting R is a complete interior line and so it must either be longer than (b_r, b_t) or has the same length and lies to the right of (b_r, b_t) . This contradicts our assumption and so R must be empty. Let b_x other than b_r and b_t be an exterior vertex at a corner of R as shown in Figure 9(d). If b_x is a terminal point, then a candidate tree is the minimal of $Tree(b_r, b_t) \cup Gap(b_t, b_x) \cup Tree(b_x, b_r)$ and $Tree(b_r, b_t) \cup Tree(b_t, b_x) \cup Gap(b_x, b_r)$. If b_x is not a terminal point, then we also need to compare the above two candidates with $Tree(b_r, b_t) \cup Tree(b_t, a_i) \cup Tree(a_{i+1}, b_r)$, where a_i and a_{i+1} are the two terminal points in $S[b_t, b_r]$ neighboring to b_x , in order to select the minimal one. Note that b_x is known for a given (b_r, b_t) . Thus, the above processing takes $O(1)$ time.

This completes the discussion for the case where (b_r, b_t) is blue. When (b_r, b_t) is non-blue, then by B6, (b_r, b_t) cannot contain any Steiner point. This makes the configuration similar to Figure 9(d). In fact, the same analysis can be carried over to construct a minimal tree for this case. Thus, the case where (b_r, b_t) can be handled in $O(1)$ time. $\square$

Consider Figure 7(b). Recall that $[b_r, b_t]$ is assumed to lie on $B_1 \cup B_2 \cup B_3$ and $[b_r, b_t] = \{b_r, b_{r+1}, \dots, b_{t-1}, b_t\}$. Let f be the interior edge properly incident to (b_r, v) . Let b_z be the exterior vertex in $[b_r, b_t]$ such that (b_z, b_t) is a horizontal grid line. Let b_p be last node in the sequence $[b_r, b_z]$ (recall that a node is a terminal point or an inner boundary corner vertex). Let b_q be the lower endpoint of the vertical boundary edge that contains b_z . Let a_p (a_q) be the terminal point closest to b_p (b_q) in $S[b_p, b_x]$ ($S[b_q, b_x]$). Let g be the interior edge immediately below f and properly incident to (v, c) .

Lemma B.1 (Lemma 7.2) *Let T be a tree of configuration C such that T cannot be converted to a Class-1 tree or a tree of configuration A by sliding edges and flipping corners. If the total length of T equals that of $\text{Tree}(b_r, b_t)$, then one of the following holds:*

1. f can be slid downward until it contains b_p , and after the sliding, b_p (a_p) is connected by a tree path in T to (b_r, v) ((v, c)) that uses f (g),
2. b_q is a terminal point, f can be slid downward until it contains b_z , and b_q (a_q) is connected by a tree path in T to (b_r, v) ((v, c)) that uses f (g).

Proof Let the interior edge f be incident to a tree edge h that lies on the boundary. If f is incident to two tree edges, then let h be the vertical one. We prove several facts from which the lemma will follow.

1. Let b_m be the exterior vertex at the intersection $f \cap h$. Then $B(b_r, b_m)$ does not monotonically decrease in both x and y coordinates and this implies that $B(b_m, b_x)$ lies on $B_1 \cup B_2$. Let b_l be the blue terminal in $[b_r, b_m]$ closest to b_r . The tree path connecting b_l to (b_r, c) cannot pass through b_r and so the path must use f . (Otherwise, the tree path lies on $B(b_r, b_l)$ as $B(b_r, b_l)$ is a shortest path between b_r and b_l . This implies that we can slide (b_r, c) to the left to become incident to blue exterior vertex in $[b_r, b_m]$ closest to b_r . But then the resulting tree is in Class-1 which contradicts our assumption.) We claim that $B(b_r, b_m)$ does not monotonically decrease in both x and y coordinates and this implies that $B(b_m, b_x)$ lies on $B_1 \cup B_2$. Suppose that $B(b_r, b_m)$ monotonically decreases in both x and y coordinates. Since b_l is connected by a tree path to (b_r, c) that uses f , we can slide f upward until it overlaps with the boundary. Since f cannot overlap with any tree edge in the sliding, f must be incident to b_l and b_r after the sliding. But the resulting tree is of configuration A which contradicts our assumption.
2. If h is horizontal, then let b_h be b_m . If h is vertical, then let b_h be the lower endpoint of the maximal sequence of vertical and adjacent tree edges containing h . Let a_h be the terminal point closest to b_h in $S[b_h, b_x]$. Then a_h is connected by a tree path to (b_r, c) that uses g . Assume to the contrary that the tree path connecting a_h to (b_r, c) uses f . Then this tree path contains $B(b_h, b_{h+1})$ as $B(b_h, b_{h+1})$ is a shortest path between b_h and a_h . If h is horizontal, then $B(b_h, b_{h+1})$ is contained in a vertical tree edge. But then by our way of choosing h , we would have chosen this vertical tree edge as h instead, a contradiction. If h is vertical, then $B(b_h, b_{h+1})$ is horizontal. We slide f downward. Since $B(b_m, b_x)$ lies on $B_1 \cup B_2$, f will overlap with $B(b_h, b_{h+1})$ and the total length will be reduced, a contradiction.
3. Suppose that b_h is not below v . If we slide f downward until f contains b_h , then f also contains the last node in the sequence $[b_r, b_z]$. Slide f downward until it contains b_h . If the right endpoint of f is v , then we are done. Suppose not. Then b_h is strictly above v . Let the right endpoint of f on (b_r, c) be u . There are two cases: (1) b_h and a_h are contained in the same boundary edge, (2) $B(b_h, a_h)$ contains one inner boundary corner vertex. In case (1), we claim that a_h must lie below v . Otherwise, we can add $B(b_h, a_h)$ to create a cycle and remove the edge joining u and v to break the cycle. Note that the total length does not increase. u becomes the vertex at a newly created complete interior corner. We flip this corner to overlap with the boundary edge containing b_r . The overlapping is a line segment and we denote it by A . Since $B(b_r, b_m)$ does not decrease monotonically in both x and y coordinates, there is an inner boundary corner vertex directly above b_h . The vertical grid line, say L , incident to this inner boundary corner vertex is blue by definition. Thus, the upper endpoint of L is blue and it lies on A . Hence, after the flipping, the tree is in Class-1 which contradicts our assumption. In case (2), we can similarly prove that a_h must lie below v and the inner boundary corner vertex contained in $B(b_h, a_h)$ is the last node in the sequence $[b_r, b_z]$. Note that f contains this inner boundary corner vertex.
4. If b_h is below v , then b_h is an outer boundary corner vertex and so it must be a terminal point. Assume to the contrary that b_h is not an outer boundary corner vertex. Then b_h and a_h must be contained in the same boundary edge. Slide f downward until it contains b_h . f cannot contain a_h ; otherwise, a cycle will be created.

Also, f cannot overlap with any tree edge during the sliding which implies that $B(b_m, a_h)$ is a vertical line segment. The fact that f can be slid along $B(b_m, a_h)$ implies that $B(b_m, a_h)$ is contained in the tree. Let u be the right endpoint of f on (b_r, c) after the sliding. We add $B(b_h, a_h)$ to create a cycle and then break the cycle by removing the edge on (b_r, c) incident to u and the right endpoint of g . Note that the total length does not increase. u will become the vertex at a newly created complete interior corner. But then this complete interior corner can be flipped to overlap with $B(b_m, a_h)$ to reduce the total length. This is a contradiction.

Now, if b_h is not below v , then we can slide f downward as stated in fact 3, and b_p is the last node in the sequence $[b_r, b_z]$ contained by f . If b_h is below v , then $b_q = b_h$ and f can be slid downward until it contains b_z . We have $a_p = a_h$ in the first case and $a_q = a_h$ in the second case. $\square$