Publication information

Title	The multiplicity one conjecture for local theta correspondences
Author(s)	Li, Jian-Shu; Sun, Binyong; Tian, Ye
Source	Inventiones Mathematicae , v. 184, (1), April 2011, p. 117-124
Version	Pre-Published version
DOI	https://doi.org/10.1007/s00222-010-0287-2
Publisher	Springer

Copyright information

© Springer-Verlag Berlin Heidelberg 2011. The final publication is available at Springer via <https://doi.org/10.1007/s00222-010-0287-2>

Notice

This version is available at HKUST Institutional Repository via <http://hdl.handle.net/1783.1/51935>

If it is the author's pre-published version, changes introduced as a result of publishing processes such as copy-editing and formatting may not be reflected in this document. For a definitive version of this work, please refer to the published version.

THE MULTIPLICITY ONE CONJECTURE FOR LOCAL THETA CORRESPONDENCES

JIAN-SHU LI, BINYONG SUN, AND YE TIAN

ABSTRACT. Over a non-archimedean local field of characteristic zero, we prove multiplicity preservation of local theta correspondences for orthogonal-symplectic dual pairs. The same proof works for dual pairs of unitary groups.

1. INTRODUCTION

Fix a non-archimedean local field k of characteristic zero. Let G be an orthogonal group $O(m)$, and let G' be a symplectic group $Sp(2n)$, both defined over k ($m, n > 0$). Then as usual, they form a reductive dual pair in the larger symplectic group $Sp(2mn)$. Denote by

$$(1) \quad 1 \rightarrow \{\pm 1\} \rightarrow \widetilde{Sp}(2mn) \rightarrow Sp(2mn) \rightarrow 1$$

the metaplectic cover of $Sp(2mn)$. Write $\widetilde{G}$ and $\widetilde{G}'$ for the double covers of G and G' , respectively, induced by the cover (1). They are centralizers of each other inside the group $\widetilde{Sp}(2mn)$.

The purpose of this note is to prove the following

Theorem A. *Let $\widetilde{G}$ and $\widetilde{G}'$ be as above. Let ω_ψ be a smooth oscillator representation of $\widetilde{Sp}(2mn)$ associated to a nontrivial character ψ of k . Then for every genuine irreducible admissible smooth representation π of $\widetilde{G}$, and π' of $\widetilde{G}'$, the inequality*

$$\dim \operatorname{Hom}_{\widetilde{G} \times \widetilde{G}'}(\omega_\psi, \pi \otimes \pi') \leq 1$$

holds.

Recall that an irreducible admissible smooth representation of $\widetilde{G}$ is said to be genuine if it does not descend to a representation of G . The same terminology applies to all double covers of groups.

Theorem A is usually called multiplicity preservation of local theta correspondences. S. Rallis call it “the Multiplicity One Conjecture” and proves it in some cases ([Ra]). It is complementary to the famous Local Howe Duality Conjecture.

2000 *Mathematics Subject Classification.* 22E35, 22E46.

Key words and phrases. Howe duality conjecture, oscillator representation, classical group, irreducible representation.

The first named author was supported in part by RGC-GRF grants 601606 and 602410 of HKSAR. The second named author was supported by NSFC grants 10801126 and 10931006.

The archimedean analog of Theorem A is proved by R. Howe in [Ho]. When the residue characteristic of k is odd, Theorem A is proved by J.-L. Waldspurger in [Wa]. For applications to number theory, it is desirable to prove multiplicity preservation over all local fields.

Remarks. (a) *The same argument as in this note proves that multiplicity preservation of local theta correspondences also hold for dual pairs of unitary groups.*

(b) *For all reductive dual pairs in the archimedean case, and all type II dual pairs in the nonarchimedean case, both multiplicity preservation and the Local Howe Duality Conjecture are proved ([Ho, Theorem 1] and [Mi, Theorem 1]).*

(c) *Unfortunately, the method of this note does not work for quaternionic type I dual pairs.*

2. PROOF OF THEOREM A

To emphasize the symmetric roles played by orthogonal groups and symplectic groups, we introduce the following notation. Let $\epsilon = \pm 1$, and let $(E, \langle \cdot, \cdot \rangle_E)$ be a finite dimensional k -vector space with a non-degenerate ϵ -symmetric bilinear form. Denote by G the automorphism group of $(E, \langle \cdot, \cdot \rangle_E)$, which is an orthogonal group or a symplectic group, depending on $\epsilon = 1$ or -1 . Following [MVW, Proposition 4.I.2], we extend G to a larger group $\check{G}$, which contains G as a subgroup of index two, and consists of pairs $(g, \delta) \in \mathrm{GL}_k(E) \times \{\pm 1\}$ such that either

$$\delta = 1 \quad \text{and} \quad g \in G,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$

Write $\epsilon' = -\epsilon$, and let $(E', \langle \cdot, \cdot \rangle_{E'})$ be a finite dimensional k -vector space with a non-degenerate ϵ' -symmetric bilinear form. Then

$$\mathbf{E} := E \otimes E'$$

is a symplectic space under the form

$$\langle u \otimes u', v \otimes v' \rangle_{\mathbf{E}} := \langle u, v \rangle_E \langle u', v' \rangle_{E'}.$$

Define the groups

$$\check{G}' \supset G' \quad \text{and} \quad \check{\mathrm{Sp}}(\mathbf{E}) \supset \mathrm{Sp}(\mathbf{E})$$

in the obvious way. Assume that both E and E' are nonzero. Then G and G' form a reductive dual pair in $\mathrm{Sp}(\mathbf{E})$.

Write

$$\mathbf{H} := \mathbf{E} \times k$$

for the Heisenberg group, where the group law is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{\mathbf{E}}).$$

The group $\check{\mathrm{Sp}}(\mathbf{E})$ acts on it as automorphisms by

$$(2) \quad (g, \delta).(u, t) := (gu, \delta t).$$

Denote the fiber product

$$\check{\mathbf{G}} := \check{G} \times_{\{\pm 1\}} \check{G}' = \{(g, g', \delta) \mid (g, \delta) \in \check{G}, (g', \delta) \in \check{G}'\},$$

which contains

$$\mathbf{G} := G \times G'$$

as a subgroup of index two. The action (2) and the homomorphism

$$(3) \quad \check{\mathbf{G}} \rightarrow \check{\mathrm{Sp}}(\mathbf{E}), \quad (g, g', \delta) \mapsto (g \otimes g', \delta)$$

induce an action of $\check{\mathbf{G}}$ on $\mathbf{H}$. This defines a semidirect product

$$\check{\mathbf{J}} := \check{\mathbf{G}} \ltimes \mathbf{H},$$

which contains

$$\mathbf{J} := \mathbf{G} \ltimes \mathbf{H}$$

as a subgroup of index two.

Let the group

$$(4) \quad \{\pm 1\} \ltimes (\check{\mathbf{G}} \times \check{\mathbf{G}})$$

act on $\check{\mathbf{J}}$ by

$$(5) \quad (\delta, \check{\mathbf{g}}_1, \check{\mathbf{g}}_2) \cdot \check{\mathbf{j}} := (\check{\mathbf{g}}_1 \check{\mathbf{j}} \check{\mathbf{g}}_2^{-1})^\delta,$$

where the semidirect product in (4) is defined by the action

$$-1 \cdot (\check{\mathbf{g}}_1, \check{\mathbf{g}}_2) := (\check{\mathbf{g}}_2, \check{\mathbf{g}}_1).$$

Note that the fibre product

$$\check{\mathbb{G}} := \{\pm 1\} \ltimes_{\{\pm 1\}} (\check{\mathbf{G}} \times_{\{\pm 1\}} \check{\mathbf{G}}) = \{(\delta, \check{\mathbf{g}}_1, \check{\mathbf{g}}_2) \mid \chi_{\mathbf{G}}(\check{\mathbf{g}}_1) = \chi_{\mathbf{G}}(\check{\mathbf{g}}_2) = \delta\}$$

is a subgroup of (4), where $\chi_{\mathbf{G}}$ is the quadratic character of $\check{\mathbf{G}}$ with kernel $\mathbf{G}$. The group $\check{\mathbb{G}}$ contains

$$\mathbb{G} := \mathbf{G} \times \mathbf{G}$$

as a subgroup of index two, and stabilizes $\mathbf{J}$ under the action (5).

We postpone the proof of the following proposition to the next section.

Proposition 2.1. *Every $\mathbb{G}$ -orbit in $\mathbf{J}$ is $\check{\mathbb{G}}$ -stable.*

Note that $\check{\mathbf{J}}$ is unimodular. Therefore by the localization principle of Bernstein and Zelevinsky ([BZ, Theorem 6.9 and Theorem 6.15 A]), Proposition 2.1 implies the following:

Corollary 2.2. *If a generalized function on $\mathbf{J}$ is $\mathbb{G}$ -invariant, then it is also $\check{\mathbb{G}}$ -invariant.*

For the usual notion of generalized functions, see [Su1, Section 2], for example. Recall the following version of the Gelfand-Kazhdan criterion (cf. [Pr]).

Lemma 2.3. *Let H be a t.d. group, i.e., a topological group which is Hausdorff, locally compact, secondly countable, and totally disconnected. Let S be a closed subgroup of H , and let σ be a continuous anti-automorphism of H . Assume that every bi- S -invariant generalized function on H is σ -invariant. Then for every irreducible admissible smooth representations π_H of H , one has that*

$$\dim \operatorname{Hom}_S(\pi_H, \mathbb{C}) \cdot \dim \operatorname{Hom}_S(\pi_H^\vee, \mathbb{C}) \leq 1.$$

Here and henceforth, we use “ $\vee$ ” to indicate the contragredient of an admissible smooth representation. A more general form of Lemma 2.3 is proved in [SZ, Theorem 2.3] for real reductive groups. The same proof works here and we omit the details.

Note that some elements of $\check{\mathbb{G}} \setminus \mathbb{G}$ act on $\mathbf{J}$ as anti-automorphisms. Therefore Corollary 2.2 and Lemma 2.3 together imply the following

Lemma 2.4. *For every irreducible admissible smooth representations $\pi_{\mathbf{J}}$ of $\mathbf{J}$, one has that*

$$\dim \operatorname{Hom}_{\mathbf{G}}(\pi_{\mathbf{J}}, \mathbb{C}) \cdot \dim \operatorname{Hom}_{\mathbf{G}}(\pi_{\mathbf{J}}^\vee, \mathbb{C}) \leq 1.$$

Now we come to the proof of Theorem A. Let $\tilde{G}$, $\tilde{G}'$ and $\widetilde{\operatorname{Sp}}(\mathbf{E})$ be double covers of G , G' and $\operatorname{Sp}(\mathbf{E})$, respectively, as in the Introduction. As in Theorem A, let π and π' be genuine irreducible admissible smooth representations of $\tilde{G}$ and $\tilde{G}'$, respectively. Recall that up to isomorphism, the oscillator representation ω_ψ is the unique genuine smooth representation of $\widetilde{\operatorname{Sp}}(\mathbf{E}) \ltimes \mathbf{H}$ which, as a representation of $\mathbf{H}$, is irreducible and has central character ψ .

It is easy to see that $\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee$ descends to an irreducible admissible smooth representation of $\mathbf{J} = (G \times G') \ltimes \mathbf{H}$ (c.f. [Su1, Lemma 5.3]). Therefore Lemma 2.4 implies that

$$(6) \quad \dim \operatorname{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \cdot \dim \operatorname{Hom}_{\mathbf{G}}(\omega_\psi^\vee \otimes \pi \otimes \pi', \mathbb{C}) \leq 1.$$

By [Su2, Theorem 1.4], the two factors in the left hand side of (6) are equal to each other. Therefore

$$\dim \operatorname{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \leq 1,$$

and consequently,

$$\dim \operatorname{Hom}_{\tilde{G} \times \tilde{G}'}(\omega_\psi, \pi \otimes \pi') \leq 1.$$

This finishes the proof of Theorem A.

3. PROOF OF PROPOSITION 2.1

We continue to use the notation of the last section. Let $\check{\mathbf{G}}$ act on $\mathbf{E}$ by

$$(7) \quad (g, g', \delta) \cdot (u \otimes u') := \delta g u \otimes g' u',$$

and act on

$$\mathbf{E}' := \operatorname{Hom}_{\mathbf{k}}(E, E')$$

by

$$((g, g', \delta) \cdot \phi)(u) := \delta g'(\phi(g^\tau u)),$$

where

$$g^\tau := \begin{cases} g^{-1}, & \text{if } \delta = 1, \\ \epsilon g^{-1}, & \text{if } \delta = -1. \end{cases}$$

Lemma 3.1. *As $\mathbf{k}$ -linear representations of $\check{\mathbf{G}}$, $\mathbf{E}$ and $\mathbf{E}'$ are isomorphic to each other.*

Proof. It is routine to check that the $\mathbf{k}$ -linear isomorphism

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{E}', \\ u \otimes u' &\mapsto (v \mapsto \langle v, u \rangle_E u') \end{aligned}$$

is $\check{\mathbf{G}}$ -intertwining. □

Denote by

$$\mathfrak{g} := \{x \in \text{End}_{\mathbf{k}}(E) \mid \langle xu, v \rangle_E + \langle u, xv \rangle_E = 0\}$$

the Lie algebra of G , and put

$$\tilde{\mathfrak{g}} := \{(x, F) \mid x \in \mathfrak{g}, F \text{ is a } \mathbf{k}\text{-subspace of } E, x|_F = 0\}.$$

Let $\check{G}$ act on $\tilde{\mathfrak{g}}$ by

$$(g, \delta) \cdot (x, F) := (\delta g x g^{-1}, gF).$$

The action of $\check{\mathbf{G}}$ on $\mathbf{E}'$ induces an action of

$$\check{G} = \check{\mathbf{G}}/G'$$

on the quotient space $G' \backslash \mathbf{E}'$.

Lemma 3.2. *There exists a $\check{G}$ -equivariant embedding from $G' \backslash \mathbf{E}'$ into $\tilde{\mathfrak{g}}$.*

Proof. Recall that the map

$$x \mapsto \langle x \cdot, \cdot \rangle_E$$

establishes a $\mathbf{k}$ -linear isomorphism from $\mathfrak{g}$ onto the space of ϵ' -symmetric forms on E . Define a map

$$\begin{aligned} \Xi : \mathbf{E}' = \text{Hom}_{\mathbf{k}}(E, E') &\rightarrow \tilde{\mathfrak{g}}, \\ \phi &\mapsto (x, F), \end{aligned}$$

where F is the kernel of ϕ , and x is specified by the formula

$$\langle \phi(u), \phi(v) \rangle_{E'} = \langle xu, v \rangle_E, \quad u, v \in E.$$

Use Witt's Theorem, one finds that two elements of $\mathbf{E}'$ stay in the same G' -orbit precisely when they have the same image under the map Ξ . Therefore Ξ descends to an embedding

$$G' \backslash \mathbf{E}' \hookrightarrow \tilde{\mathfrak{g}}.$$

Finally, one checks easily that this embedding is $\check{G}$ -equivariant. □

The following lemma is stated in [MVW, Proposition 4.I.2]. We will not provide a proof here.

Lemma 3.3. *For every $(x, F) \in \tilde{\mathfrak{g}}$, there is an element $(g, -1) \in \check{G}$ such that*

$$gxg^{-1} = -x \quad \text{and} \quad gF = F.$$

In other words, every element of $\tilde{\mathfrak{g}}$ is fixed by an element of $\check{G} \setminus G$. Therefore every G -orbit in $\tilde{\mathfrak{g}}$ is $\check{G}$ -stable. Now Lemma 3.2 implies that every G -orbit in $G' \setminus \mathbf{E}'$ is $\check{G}$ -stable, or equivalently, every $\mathbf{G}$ -orbit in $\mathbf{E}'$ is $\check{\mathbf{G}}$ -stable. Finally, by Lemma 3.1, we have

Lemma 3.4. *Every $\mathbf{G}$ -orbit in $\mathbf{E}$ is $\check{\mathbf{G}}$ -stable.*

Now we are ready to prove Proposition 2.1. Note that every $\mathbb{G} = \mathbf{G} \times \mathbf{G}$ -orbit in $\mathbf{J}$ intersect the subgroup $\mathbf{H}$, and the subgroup

$$\check{\mathbb{G}} = \{\pm 1\} \times_{\{\pm 1\}} (\Delta(\check{\mathbf{G}})) \quad \text{of} \quad \check{\mathbb{G}} = \{\pm 1\} \ltimes_{\{\pm 1\}} (\check{\mathbf{G}} \times_{\{\pm 1\}} \check{\mathbf{G}})$$

stabilizes $\mathbf{H}$, where “ Δ ” stands for the diagonal group. Therefore in order to prove Proposition 2.1, it suffices to show that every $\mathbf{G}$ -orbit in $\mathbf{H}$ is $\check{\mathbf{G}}$ -stable. Note that as a $\check{\mathbf{G}}$ -space,

$$\mathbf{H} = \mathbf{E} \times \mathbf{k},$$

where $\mathbf{E}$ carries the action (7), and $\mathbf{k}$ carries the trivial $\check{\mathbf{G}}$ -action. We finish the proof by Lemma 3.4.

REFERENCES

- [BZ] I. N. Bernstein and A. V. Zelevinsky, *Representations of the group $GL(n, F)$ where F is a non-archimedean local field*, Russian Mathematical Surveys (1976) 31, no. 3, 1-68.
- [Ho] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. 2 (1989), no. 3, 535-552.
- [Mi] A. Minguez, *Correspondance de Howe: paires duales de type II*, Ann. Sci. Ec. Norm. Sup., (serie 4) 41 (2008), 717-741.
- [MVW] C. Moeglin, M.-F. Vigneras, and J.-L. Waldspurger, *Correspondence de Howe sur un corp p -adique*, Lecture Notes in Math. 1291, Springer, 1987.
- [Pr] D. Prasad, *Trilinear forms for representations of $GL(2)$ and local ϵ -factors*, Compositio Math 75 (1990), no. 1, 1-46.
- [Ra] S. Rallis, *On the Howe duality conjecture*, Compositio Math. 51 (1984), 333-399.
- [Su1] B. Sun, *Multiplicity one theorems for Fourier-Jacobi models*, arXiv:0903.1417.
- [Su2] B. Sun, *Dual pairs and contragredients of irreducible representations*, arXiv:0903.1418. (To appear in Pacific J. Math.)
- [SZ] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, arXiv:0903.1409.
- [Wa] J.-L. Waldspurger, *Démonstration d’une conjecture de dualité de Howe dans le cas p -adique, $p \neq 2$* , Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, 267-324.

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, HONG KONG

E-mail address: `matom@ust.hk`

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, P.R. CHINA

E-mail address: `sun@math.ac.cn`

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, P.R. CHINA

E-mail address: `ytian@math.ac.cn`