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Unique critical state characteristics in granular media considering fabric anisotropy

J. ZHAO* and N. GUO*

The concept of the critical state in granular soils needs to make proper reference to the fabric structure that develops at critical state. This study identifies a unique property associated with the fabric structure relative to the stresses at critical state. A unique relationship between the mean effective stress and a fabric anisotropy parameter, \( K \), defined by the first joint invariant of the deviatoric stress tensor and the deviatoric fabric tensor, is found at critical state, and is path-independent. Numerical simulations using the discrete-element method under different loading conditions and intermediate principal stress ratios identify a unique power law for this relationship. Based on the findings, a new definition of critical state for granular media is proposed. In addition to the conditions of constant stress and unique void ratio required by the conventional critical state concept, the new definition imposes the additional constraint that \( K \) reaches a unique value at critical state. A unique spatial critical state curve in the three-dimensional space \( K-e-p' \) is found for a granular medium, the projection of which onto the \( e-p' \) plane turns out to be the conventional critical state line. The new critical state concept provides an important reference state for a soil to reach, based on which the key concepts in the constitutive modelling of granular media, including the choice of state parameters, dilatancy relation and non-coaxiality, are reassessed, and future exploratory topics are discussed.

KEYWORDS: anisotropy; discrete-element modelling; fabric/structure of soils; plasticity

INTRODUCTION

The critical state theory (CST) developed by Roscoe et al. (1958) and Schofield & Wroth (1968) laid the foundation for critical state soil mechanics. Fundamental to the theory is the concept of critical state that identifies an ultimate state for a soil to converge under sustained shear. Critical state in granular soils refers to a state of continuous shear deformation with a constant volume under constant stress. In essence, at critical state, a soil may reach a constant stress ratio \( \eta = q/p' = M \) (where \( p' \) and \( q \) are the commonly referred mean effective stress and the deviatoric stress respectively, and \( M \) denotes a material coefficient that may depend on the shear mode – for example, the Lode angle or intermediate principal stress ratio – and a unique critical void ratio \( e_c \). The classical definition of critical state emphasises fabric isotropy (the void ratio), but lacks a proper reference to fabric anisotropy. Experimental and numerical studies have indicated that the behaviour of a granular soil under shear is predominantly anisotropic. Under sustained shear, the induced plastic flow deformation will mobilise particles in a granular assembly to adjust themselves by sliding and rolling to provide better support for the external load, which naturally leads to the formation of an anisotropic fabric structure to serve this role optimally. The anisotropic fabric structure evolves steadily with the loading process. Indeed, early micromechanical studies suggested that the macroscopic strength of a granular material is strongly correlated with the degree of anisotropy in the fabric structure during almost all stages of the loading history (e.g. Rothenburg & Bathurst, 1989). It has been shown that the anisotropic fabric structure sustains the major-
there has been significant progress in the development of advanced experimental techniques for probing the grain-scale information in granular media, such as X-ray microtomography and MRI imaging (Ng et al., 1997; Mueth et al., 2000; Hall et al., 2010; Hasan & Alshibli, 2010, 2012), it remains a great practical challenge for experimenters to effectively explore and accurately quantify the intricate characteristics of the fabric structure in granular media at the particle scale, particularly when in a critical state. Numerical tools are the primary option for conducting such investigations: for instance, direct simulation of the grain system is possible using the DEM. Although not perfect, the DEM may help to extract useful information on the grain scale, including information relevant to the fabric structure at critical state. In this paper, a three-dimensional DEM is used to explore the signature properties of the critical state in a granular material by taking the critical fabric anisotropy into consideration.

METHODOLOGY AND APPROACH

Sample preparation and loading paths

The three-dimensional DEM code used in this study uses a linear force–displacement contact law in conjunction with Coulomb’s friction law, which governs the sliding friction (Abe et al., 2004; Guo & Zhao, 2013). The normal stiffness and tangential stiffness are set to be \( k_n/r = k_t/r = 100 \text{ MPa} \), where \( r \) denotes the equivalent radius of two contacted particles. The interparticle friction coefficient in Coulomb’s law adopts the value \( \mu = 0.5 \), which is typical for quartz sand. Around 32,000 polydisperse spherical particles with radii ranging from 0.2 mm to 0.6 mm are randomly generated in a cubic container with rigid frictionless walls. The particle-size distribution of an assembly is approximated to that of Toyoura sand (see Guo & Zhao (2013) for a detailed description of the approximating method). Special techniques for staged isotropic consolidation have been developed to produce random assemblies with different initial void ratios (Guo & Zhao, 2013). As all particles are randomly generated and isotropically consolidated, the influence of the initial fabric anisotropy on the shear response is excluded from consideration (the significance of which will be discussed later). All obtained samples are then monotonically sheared in two loading conditions (drained and undrained), which are commonly treated in soil mechanics. To conduct the tests in the drained condition, the horizontal pressures on the four vertical walls of the cube are kept constant during the shear using a servo-control technique. In the undrained tests, only dry particles are considered: hence the undrained condition is simulated in an approximate sense by imposing a constant-volume constraint on the sheared sample (see also Yimsiri & Soga (2010)). During the undrained shear, the horizontal strain is continuously adjusted according to the applied vertical compressive force, while the total volume of the assembly is maintained constant. To investigate the influence of the loading path on the final observations, various constant-b tests (see also Barreto & O’Sullivan (2012)) have been explored, where \( b = (\tau_2 - \tau_3)/(\sigma_1 - \sigma_3) \) is the intermediate principal stress ratio. Five different cases of \( b \) were investigated: \( b = 0, 0.25, 0.5, 0.75 \) and 1. Notably, \( b = 0 \) and \( b = 1 \) correspond to conventional triaxial compression and triaxial extension respectively. To gain more confidence in the final data, some supplementary constant-p’ tests were also conducted.

Stress tensor and fabric tensor

The following definition of a stress tensor proposed by Christoffersen et al. (1981) is applied to quantify the macroscopic response of a DEM assembly.

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_1 & \sigma_2 & \sigma_3 \\
\sigma_2 & \sigma_4 & \sigma_5 \\
\sigma_3 & \sigma_5 & \sigma_6
\end{bmatrix}
\]

where \( \sigma_1, \sigma_2, \ldots, \sigma_6 \) are the principal stresses.

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

where \( u_i \) is the displacement in the \( i \)-direction.

\[
\varepsilon_\text{strain} = \sqrt{\frac{1}{2} \left( \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \right)}
\]

is the strain energy density function.
\[ \sigma_{ij} = \frac{1}{V} \sum_{c \in N_c} \mathbf{f}_c^* d \mathbf{n}_c \]  

(1)

where \( V \) is the total volume of the assembly, \( N_c \) is the total number of contacts, \( \mathbf{f}_c^* \) denotes the contact force at a contact, and \( \mathbf{n}_c \) defines the branch vector joining the centres of two contacted particles. The mean effective stress and deviatoric stress can then be determined by \( p' = \sigma_{ij} - \delta_{ij} p \) (\( \delta_{ij} \) is the Kronecker delta). As the treatment involves cubic samples confined by rigid walls, the axial strain and the volumetric strain can be approximately defined by the displacement of the boundary walls.

To characterise the fabric structure, a suitable variable is needed to quantify the fabric anisotropy. Among the various definitions of fabric tensor (e.g. Oda, 1972a, 1982; Satake, 1982; Kanatani, 1984; Baji, 1996; Li & Li, 2009), the contact-normal-based proposition by Satake (1982) and Oda (1982) is chosen here.

\[ \phi_{ij} = \int_\Theta E(\Theta) n_i n_j d\Theta \]  

(2)

where \( n \) is the unit vector along the normal direction of the contact plane; \( \Theta \) characterises the orientation of \( n \) relative to the global coordination system; and \( E(\Theta) \) is the distribution probability density function (PDF). In most cases it is sufficient to apply the second-order Fourier expansion of \( E(\Theta) \) to characterise the contact normals (Ouadfel & Rothenburg, 2001),

\[ E(\Theta) = \frac{1}{4\pi} (1 + F_{ij} n_i n_j) \]  

(3)

where the second-order anisotropic fabric tensor \( F_{ij} = 15/2(3\phi_{ij} - 1/3\phi) \) is deviatoric and symmetric, and can be used to characterise the fabric anisotropy in the assembly. In practice, \( \phi_{ij} \) (and hence \( F_{ij} \)) can be estimated from the discrete data of a granular assembly by

\[ \phi_{ij} = \frac{1}{N_c} \sum_{c \in N_c} n_i^c n_j^c \]  

where \( n^c \) denotes the unit contact normal vector.

**Joint invariants of the stress tensor and the fabric tensor**

Owing to the deviatoric nature of \( F_{ij} \), its three invariants can be defined as follows.

\[ J_1^F = F_{ii} = 0 \]  

\[ J_2^F = \frac{1}{2} F_{ij} F_{ji} \]  

\[ J_3^F = \frac{1}{2} F_{ij} F_{jk} F_{ki} \]  

(4)

According to the representation theory (Wang, 1970; Spencer, 1971), any scalar-valued isotropic function of two second-order tensors can be expressed in terms of their own invariants together with up to four irreducible joint invariants between them. Consider the stress tensor defined in equation (1) and the fabric tensor \( F_{ij} \); a scalar-valued function \( f \) of the two tensors can be expressed as

\[ f = f(\sigma_{ij}, F_{ij}) \]

\[ = f(I_1, J_{2D}, J_{3D}, J_1^F, J_2^F, J_3^F, K_1, K_2, K_3, K_4) \]

where \( I_1, J_{2D}, J_{3D} \) are respectively the first invariant of the stress tensor and the second and third invariants of the deviatoric stress tensor \( \delta_{ij} \); \( K_i \) (for \( i = 1, 2, 3, 4 \)) denotes the four joint invariants between the deviatoric stress tensor and the fabric tensor, given by

\[ K_1 = K = s_{ij} F_{ij} \]  

\[ K_2 = s_{ij} F_{jk} F_{ki} \]  

\[ K_3 = s_{ijk} s_{jkl} \]  

\[ K_4 = s_{ijk} s_{jkl} F_{ij} F_{kl} \]  

(6)

In the following, the first joint invariant \( K_1 \) is replaced by \( K \) for convenience.

**RESULTS AND OBSERVATIONS**

Over 80 numerical samples were sheared from different initial states (i.e. in terms of density and initial confining pressure) to a critical state under various monotonic loading paths. In our numerical simulations, all samples were found to reach a relatively steady state with constant stress and constant volume beyond an axial strain level of around 40%, which satisfies the classic description of a critical state. The soil state beyond this point is regarded as a critical state. Nevertheless, there are certain degrees of fluctuation in the soil response after this strain level. To minimise possible deviations caused by fluctuations, the relevant quantities (e.g. \( F_{ci}, K_c \)) were averaged over a sustained stage of deformation, for example for an axial strain level ranging between 40% and 50%. The averaging technique appeared to be fairly effective, and led to relatively stable and consistent quantifications of critical states.

**Critical void ratio**

The critical void ratio was correlated with the mean effective stress \( p' \) for all samples under investigation, as shown in Fig. 2. All data points were found to collapse to the following general form (cf. Li, 1997; Li et al., 1999).

\[ e_c = c_1 - \lambda_c \left( \frac{p'}{p_a} \right)^\xi \]  

(7)

where \( c_1 \) denotes the critical void ratio at \( p' = 0 \), \( \lambda_c \) and \( \xi \) are material constants, and \( p_a = 101 \text{kPa} \) is the atmospheric pressure. Our simulated data fit the expression in equation (7) well with \( c_1 = 0.663 \), \( \lambda_c = 0.008 \) and \( \xi = 1.0 \), which represents a linear relation. (The data are presented in Fig. 2 in a semi-log scale rather than a natural scale to render the
data points in an even distribution over the chosen stress range.) Notably, a similar linear relationship was previously reported by Been et al. (1991) for Erksak sand, and by Ishihara (1993) for Toyoura sand. Note that the loading paths (e.g. with varying $b$), do not affect the above relationship. Evidently, the study identifies a unique critical state line in the $c$–$p'$ plane for sand, which appears to support the general view taken by Been et al. (1991) and Ishihara (1993) on the uniqueness of the CSL (see also Jefferies & Been, 2006).

**Critical fabric anisotropy**

It is interesting to examine whether similar unique features can be identified for fabric anisotropy at critical state. The second invariant of $F_{ij}$

$$F_c = \sqrt{3I^2_{12}I^2_{23}I^2_{31}}/2$$

is first selected to represent the degree of critical fabric anisotropy. The correlation between $F_c$ and $p'$ at critical state is plotted in Fig. 3. Interestingly, for each individual case of $b$, a power-law dependence of $F_c$ on the critical mean effective stress $p'$ is observed (cf. Guo & Zhao (2013) for the triaxial compression cases)

$$F_c = m_b \left(\frac{p'}{p_a}\right)^\zeta$$

where $m_b$ is a parameter dependent on the Lode angle or $b$ (e.g. $m_0 = m_b(b)$). The numerical results indicate $\zeta = -0.14$ to $-0.09$ for $b \in [0, 1]$. Evidently, according to the present definition of ‘fabric tensor’, a unique critical fabric structure independent of the loading path is not attainable.

Although not presented here, the third invariant of $F_{ij}$ in a critical state, $J^3$, was also found to be non-unique. To explore whether the invariant functions of the three invariants lead to unique characterisations, the following function, which borrows the form of Lade’s failure criterion, was also examined (cf. Thornton, 2000; Thornton & Zhang, 2010).

$$\eta^* = \frac{\left(\frac{I^3}{I^1}\right)^9}{2I^1 I^2 I^3 - 3I^3}$$

where $I^i_j$ ($i = 1, 2, 3$) are the three invariants of the original fabric tensor $\phi_{ij}$ (not $F_{ij}$). Using this newly defined $\eta^*$, the results of various constant-$b$ cases in critical states are replotted in Fig. 4. Although the use of an invariant function in the form of equation (9) considerably reduces the range of variation for all data, it still cannot unify all cases uniquely. Several different forms of invariant function (e.g. the Matsuoka–Nakai criterion-like invariant) were tried, and the observations are more or less similar to $\eta^*$. The examination using the usual measures of fabric tensor, as shown above, suggests that the critical fabric structure in a granular material is not unique (or loading path dependent).

**Joint invariants at critical state**

The property of critical fabric anisotropy is not unique. Meanwhile, it is known that the critical stress ratio (e.g. $q/p'$) varies with the loading path. As shown by several past studies, the fabric anisotropy is intimately related to the stress state (e.g. Oda et al., 1985; Thornton & Zhang, 2010). Hence it is inappropriate to quantify their critical state properties separately. In this case, the joint invariants of the two tensors may offer a more accurate characterisation of a soil in a critical state. To verify this, in Figs 5 and 6 the correlations between the third and the fourth joint invariants, $K_3$ and $K_4$, and $p'$ exhibit trends similar to those of the second and first joint invariants respectively, which are not presented here. The correlation between $K$ and $p'$ is presented comparatively in both a natural scale and a log–log scale in Fig. 5. A striking observation is that the $K$ value at critical state, $K_c$, can be uniquely correlated with the critical mean effective stress $p'$ by the following power law

$$K_c = 0.41p'^{0.894}$$

Considering the great variety of sands, a more general power law may be postulated

$$K_c = ap'^\zeta$$

where $a$ and $\zeta$ are material constants. In contrast to the case of $K$, the correlation between $K_3$ (or $K_4$) and $p'$ at critical state, as shown in Fig. 6, demonstrates a clear path-dependent feature. Consequently, $K_c$ appears to be a better indicator of the compatibility of the critical stress with the fabric structure, and may be adopted to characterise the unique property of the critical state.

Based on equation (6), a pure measure of the relative orientation of the two tensors, $A'$, can be further defined:
that is, $A' = n_{\|}^T n_{\|}$, where $n_{\|}$ denotes the deviatoric stress direction and $n_{\|}$ is the fabric direction. Note that $A'$ is indeed a normalised quantity of the anisotropic variable $A$, as defined in Li & Dafalias (2012). The evolution of $A'$ during the shearing process for five samples with different $b$ is shown in Fig. 7. As shown, the stress and the fabric tensor tend to become coaxial quickly upon shear, and $A'$ always approaches unity or a value very close to unity shortly after the application of shear, and then stays at this value before reaching the critical state. The observation provides clear evidence that the coaxiality of the deviatoric stress and the fabric anisotropy is indeed a very special property of the critical fabric structure, which is consistent with Li & Dafalias (2012).

**State surface of $K_c$ in the deviatoric plane**

The above observation can be further verified by visualising the state surfaces for the various quantities concerned at critical state. This is inspired by the study conducted by Thornton & Zhang (2010). Following their approach, the state surfaces for both the critical state stresses and the critical fabric anisotropy are plotted in a three-dimensional deviatoric plane, in a manner analogous to that for a failure criterion. Fig. 8(a) shows the critical stress surface and critical fabric anisotropy in the deviatoric plane obtained by different constant-$b$ tests at a pressure level of around 1000 kPa. It is clearly observable that the critical state stresses form a smooth triangular-shaped surface, reminiscent of a Lade’s criterion surface, whereas the critical fabric anisotropies form a surface with an inverted smoothed triangular shape, reciprocal to that of the critical state stresses. Thornton & Zhang (2010) observed similar behaviour for these quantities during the loading course of a granular assembly en route to a critical state. Yimsiri & Soga (2010) also observed a larger final fabric anisotropy under triaxial extension than under triaxial compression, which is consistent with the authors’ observation in Fig. 8(a).

Evidently, neither the critical state stress nor the critical fabric anisotropy has a circular state surface in the deviatoric plane, which explains their dependence on the loading path. Nevertheless, noting the complementary nature of the two surfaces, it is apparent that proper combinations of the two may lead to a circular state shape. The definition in equation (6) for the first (or the fourth) joint invariant offers one such combination. A further plot of $K_c$ indeed confirms this expectation. As shown in Fig. 8(b), $K_c$ does display a well-rounded circular shape in the deviatoric plane, which is independent of the Lode angle. This feature renders it suitable for use as a reference state for soil state characterisation.

The changes in the size and shape of the critical state yield surface, fabric anisotropy and $K_c$ at six pressure levels (80 kPa, 300 kPa, 500 kPa, 700 kPa, 1000 kPa and 2000 kPa) are further plotted in Fig. 9. As shown in Figs 9(a) and 9(b),
lower pressure levels lead to apparently smaller critical state yield surfaces, whereas the corresponding state surfaces for critical fabric anisotropy appear to be larger. Although the shapes of the critical state yield surface and the fabric anisotropy at each pressure level generally resemble those at \( p_c = 1000 \text{ kPa} \), at lower pressures (e.g. \( p_c \approx 80 \text{ kPa} \)) both the critical state yield surface and the state surface for \( F_c \) are more rounded than they appear to be at higher pressures.

In particular, if \( r_M = M_c/M_e \) denotes the ratio between the critical stress ratios under triaxial compression and triaxial extension, it is evident from Fig. 9(a) that this ratio increases from around 1 to 1.2 when the critical pressure increases from 80 kPa to 2000 kPa. This ratio appears to be less sensitive to pressure in the low-pressure regime, but seems to become more sensitive when the pressure is high. A similarly defined ratio for \( F_c \) demonstrates a similar trend, except that it decreases from 1 to around 0.84 as the pressure increases (see Fig. 8(b)). In contrast, in all cases \( K_c \) is found to be a circle similar to that in Fig. 8(b), albeit with different radii. If a normalised measure, \( K_c = K_c / p_c^{0.894} \), is further defined to rescale the data, all surfaces of \( K_c \) are found to collapse approximately to a circle \( K_c = \text{constant} \) (around 0.41 \pm 0.01 from the obtained data), as shown in Fig. 9(c), albeit with some minor deviations.

**Unique critical state curve in \( K_c - e_c - p'_c \) space**

In view of the observed unique dependence of both \( e_c \) and \( K_c \) on \( p'_c \), it is evident that the three quantities are indeed
intercorrelated in a unique manner. The unique relationship is presented in the form of a spatial curve in $K_c-e_c-p'_c$ space, as shown in Fig. 10. The two critical lines in Figs 2 and 5 are its projections in the $e_c-p'_c$ plane and $K_c-p'_c$ planes respectively. This new critical state curve nicely unifies the classic critical state concept with the case of fabric anisotropy. Note that the critical state line in $e-p'_c-\phi$ space presented by Roscoe et al. (1963) (Fig. 6 therein) is indeed a special case of our critical state line when $F_i \equiv 1$ and $A' = 1$.

**DISCUSSION AND OUTLOOK**

It is instructive to discuss the significance of these new findings for some fundamental concepts and methodologies in theoretical soil mechanics. Only a brief discussion is devoted to each topic, owing to space limitations.

**Critical state conditions**

It is clear from the obtained findings that a complete picture of the critical state for granular materials needs to consider both the unique isotropic characteristics of the critical state and the critical fabric structure relevant to the critical stress. Following a method similar to that of Li & Dafalias (2012), a third condition on the anisotropic variable $K$ is introduced into the classic critical state concept to furnish a new definition of anisotropic critical state as

$$
\eta = \frac{q_c}{p'_c} = M(b)
$$

$$
e = e_c = \dot{e}_c(p')
$$

$$K = K_c = K_c(p')
$$

(12)

Although the specific loading and fabric evolution law may be path dependent, the critical state conditions in equation (12) set a unique ultimate goal (or compass) for both the fabric anisotropy and the critical state, which requires additional reference to the state of the fabric.

**Anisotropic state parameter**

The isotropic state parameter defined by Been & Jefferies (1985), $\psi = e - e_c$, reflects at best an isotropic soil state, which requires additional reference to the state of the fabric anisotropy. Based on the unique property shown by $K_c$, the following pair of state parameters is proposed to characterise the state in sand (note that $K_c > 0$)

$$
\psi = e - e_c
$$

$$\phi = \frac{K_c}{K_c}
$$

(13)

Evidently, the critical state values for the two state parameters are $\psi = 0$, $\phi = 1$. To evaluate $\phi$, it is important to recognise that the critical $K_c$ might not necessarily be a maximum. Equation (13) borrows the definitions of $\psi$ and $\eta$ from Li & Dafalias (2012) (equations (12)–(15) therein).

**Stress dilatancy**

The experimental data indicate that the dilatancy of sand is influenced by fabric anisotropy. With the help of the state parameters defined in equation (13), the following state-dependent dilatancy relationship is suggested (cf. Li & Dafalias, 2000)

$$D = D(\psi, \phi, \eta, C)
$$

(14)

where $C$ denotes a collection of intrinsic material constants. There may be more than one way to define the specific form of $D$. A specific example is demonstrated here. Assume an additive form of the state parameters defined in equation (13)

$$\theta = \psi + k[1 - (n^L : n')\phi]
$$

(15)

where $k$ is a material constant, and $n^L$ is a unit tensor denoting the loading direction (e.g. the plastic strain rate direction; Li & Dafalias, 2012). The following general dilatancy relation may be proposed

$$D = D(X(\theta) - Y(\eta))
$$

(16)

where $X$ and $Y$ are scalar-valued functions of $\theta$ and $\eta$ respectively. A modified form of $D$ from the expression in Li & Dafalias (2012) may be suggested

$$D = d(M^{\text{add}} - \eta)
$$

(17)

where $m$ and $d$ are positive material constants. The following observations are apparent from equation (17) (cf. Li & Dafalias, 2012).

(a) At critical state, $\psi = 0$ and $\phi = 1$, $\theta = 0$, $\eta = M$: thus $D = 0$.

(b) During flow liquefaction, $\theta$ approaches 0 and $\eta = M$, so that $D = 0$. However, as $\psi \neq 0$ and $\phi \neq 1$, it differs essentially from the critical state.

(c) The attainability of the phase transformation state as defined by Ishihara et al. (1975) for different sands can be easily explained, with $D$ jointly controlled by $\phi$ and $\theta$ as well as by $\eta$.

(d) When the loading direction changes so abruptly (e.g. from $n^L$ to $-n^L$) that both the fabric anisotropy and the stress direction remain unchanged, $K$ stays at its original value. But according to equation (15) $n^L : n'$ may change its sign (e.g. from positive to negative), resulting in a very large value for $k[1 - (n^L : n')\phi]$ such that $\theta > 0$ and $D > 0$. This may naturally lead to cyclic contraction.

**Non-coaxiality**

The new findings may also facilitate the modelling of non-coaxiality between the stress and the plastic strain.
increment, as observed by Gutierrez et al. (1993), Yoshimine et al. (1998) and others. The plastic strain increment is defined according to the associated flow rule

$$\text{d}e_p^i = \langle \text{d}L \rangle \frac{\partial f}{\partial \sigma^i}$$

(18)

where $\langle \text{d}L \rangle$ denotes a non-negative loading multiplier. Assume a fabric-dependent yield function with the form

$$f = f(\sigma^i, F^i, H, K)$$

(19)

where $H$ is a hardening parameter. The use of equation (19) in equation (18) leads to

$$\text{d}e_p^i = \langle \text{d}L \rangle \frac{\partial f}{\partial \sigma^i} + \frac{\partial f}{\partial K} \frac{\partial K}{\partial \sigma^i}$$

(20)

where $f$ denotes the terms in the yield function without the involvement of $K$. The second term in the right-hand bracket of equation (20), $(\partial f/\partial K)(\partial K/\partial \sigma^i)$, may account for the non-coaxial behaviour observed in granular materials. A rather significant difference between the fabric orientation and the loading direction in the early stage of loading may induce this term to deviate substantially from the stress direction, thus contributing to the overall non-coaxial plastic strain increment. At a relatively large strain level, when $o_{ij}$ and $F^i$ become coaxial and compatible with each other, this term will become more coaxial with $o_{ij}$, as will the overall plastic strain increment $\text{d}e_p^i$ with respect to $o_{ij}$.

Other relevant issues

There are other noteworthy issues relevant to the observations found in the study, including initial anisotropy, particle crushing and strain localisation. All the observations presented so far have been based on consideration of initially isotropic sand samples. The influence of initial (or inherent) anisotropy can be investigated further using non-spherical DEM particles. Nevertheless, it is believed that the shearing process will totally destroy all previous memories of granular media, including initial anisotropy, such that the critical state will not be affected by the initial state (see also Yioumisir & Soga, 2010). Meanwhile, real experimental tests on sand involve some level of particle crushing, which was not considered in this study. It is expected, however, that significant particle crushing will lead to an essentially quite different material in the final stage from that at the beginning. Hence it is irrelevant to discuss the uniqueness of the critical state in this case. Regarding strain localisation, this effect has rarely been observed in the present authors’ DEM simulations, probably because of the use of rigid walls and free-rolling particles. It remains arguable whether it is still valid to discuss the critical state as being homogeneous, as envisaged by Roscoe et al. (1958), in the presence of strain localisation in a soil sample.

CONCLUSIONS

Motivated by the anisotropic critical state theory recently proposed by Li & Dafalias (2012), this study explored the characteristics of critical states in granular materials under the influence of fabric anisotropy. Using a 3D DEM, granular assemblies with different initial void ratios and confining pressures were sheared to a critical state under both drained and undrained loading conditions, and with different intermediate principal stress ratios. A number of novel observations were identified from the results.

(a) The critical state in granular media can be uniquely characterised. Rather than referring to an isotropic state, it is always associated with a fabric structure compatible with the critical stress state.

(b) The critical fabric anisotropy is not unique, but depends on the specific loading path. It is strongly related to the critical stress state, and cannot be separated as an individual or unique reference for the soil state.

(c) The first joint invariant of the stress tensor and the fabric tensor, $K$, is uniquely related to the pressure level at critical state, and the correlation is path-independent.

(d) The correlations among $K, e$ and $p^*$ at critical state can be visualised by a unique spatial critical state curve in $K–e–p^*$ space. The projection of this curve onto the $e–p^*$ plane is the critical state plane defined by classic critical state theory. A new definition of the critical state is proposed by incorporating the additional unique condition of $K_c$.

(e) The critical $K_c$ can be further normalised to be a constant that is independent of either the loading path or the pressure level.

In light of the new findings, future challenges and opportunities for the development of theoretical soil mechanics are discussed above, with relevance to such topics as the choice of state parameters, the definition of the dilatancy relation, the modelling of non-coaxiality, and the quantification of the critical state in the presence of strain localisation, particle crushing and initial anisotropy.

NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$A$</td>
<td>fabric anisotropic variable used by Li &amp; Dafalias (2012)</td>
</tr>
<tr>
<td>$A'$</td>
<td>measure of relative orientation between fabric and stress direction</td>
</tr>
<tr>
<td>$A_c$</td>
<td>critical fabric anisotropic variable used by Li &amp; Dafalias (2012)</td>
</tr>
<tr>
<td>$b$</td>
<td>intermediate principal stress ratio</td>
</tr>
<tr>
<td>$C$</td>
<td>material constant in equations (14) and (17)</td>
</tr>
<tr>
<td>$D$</td>
<td>dilatancy</td>
</tr>
<tr>
<td>$d$</td>
<td>material constant in equations (14) and (17)</td>
</tr>
<tr>
<td>$d'$</td>
<td>branch vector and its component in the $i$th direction</td>
</tr>
<tr>
<td>$E(\Theta)$</td>
<td>contact normal probability density function</td>
</tr>
<tr>
<td>$e$</td>
<td>void ratio</td>
</tr>
<tr>
<td>$e_c$</td>
<td>critical void ratio</td>
</tr>
<tr>
<td>$e_f$</td>
<td>critical void ratio at zero pressure</td>
</tr>
<tr>
<td>$F$</td>
<td>degree of fabric anisotropy</td>
</tr>
<tr>
<td>$F_c$</td>
<td>critical fabric anisotropy</td>
</tr>
<tr>
<td>$F_y$</td>
<td>deviatoric fabric tensor</td>
</tr>
<tr>
<td>$f$</td>
<td>yield function</td>
</tr>
<tr>
<td>$f_c$</td>
<td>contact force vector and its component in the $i$th direction</td>
</tr>
<tr>
<td>$H$</td>
<td>hardening parameter in equation (19)</td>
</tr>
<tr>
<td>$I_{1}$</td>
<td>three invariants of stress tensor</td>
</tr>
<tr>
<td>$I_{2}$</td>
<td>three invariants of original fabric tensor</td>
</tr>
<tr>
<td>$I_{3}$</td>
<td>three invariants of deviatoric fabric tensor</td>
</tr>
<tr>
<td>$K_{(K)}$, $K_{(K)}$, $K_{(K)}$, $K_{(K)}$</td>
<td>four joint invariants between deviatoric stress tensor and fabric tensor</td>
</tr>
<tr>
<td>$K_c$</td>
<td>critical fabric anisotropy parameter</td>
</tr>
<tr>
<td>$k$</td>
<td>material constant in equation (13)</td>
</tr>
<tr>
<td>$k_n$, $k_t$</td>
<td>normal and tangential stiffness</td>
</tr>
<tr>
<td>$\langle \text{d}L \rangle$</td>
<td>non-negative loading multiplier</td>
</tr>
<tr>
<td>$M$, $M_{(K)}$</td>
<td>critical stress ratio</td>
</tr>
<tr>
<td>$m$</td>
<td>material constant in equations (14) and (17)</td>
</tr>
<tr>
<td>$N_c$</td>
<td>total contact number within volume</td>
</tr>
<tr>
<td>$n$, $n_i$</td>
<td>unit contact normal vector and its component in the $i$th direction</td>
</tr>
</tbody>
</table>
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\( \mathbf{r} \) unit tensor denoting the fabric direction in equation (15)
\( \mathbf{p} \) unit tensor denoting the loading direction in equation (15)
\( \mathbf{n} \) unit-norm deviatoric fabric tensor used by Li & Dafalias (2012)
\( \psi \) unit-norm deviatoric stress tensor used by Li & Dafalias (2012)
\( \hat{\mathbf{p}} \) mean effective stress
\( \rho_a \) atmospheric pressure
\( q \) deviatoric stress
\( r^2 \) regression coefficient of determination
\( s_y \) deviatoric stress tensor
\( \gamma \) sample volume
\( \alpha \) material constant in equation (11)
\( \delta \) exponent in power law for fanning \( q \rightarrow q' \) at critical state
\( \delta_{ij} \) Kronecker delta
\( \varepsilon \) accumulated axial strain
\( \phi_p \) plastic strain increment
\( \eta \) material parameter in equation (8)
\( \eta^p \) Lade-like scalar function of \( \mathbf{I}^p \), \( \mathbf{I}^2 \) and \( \mathbf{I}^3 \)
\( \theta \) overall state parameter
\( \lambda \) material constant in equation (7)
\( \lambda^c \) material constant in equation (7)
\( \sigma_1, \sigma_2, \sigma_3 \) the three principal stresses
\( \sigma_c \) stress tensor
\( \phi \) anisotropic state parameter
\( \phi_f \) fabric tensor
\( \psi \) isotropic state parameter

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