Competition of Multi-Agent Systems: Analysis of a Three-Company Econophysics Model *

FU Xiu-Jun(傅秀军)^{1,2**}, Szeto K Y(司徒国业)²

¹Department of Physics, South China University of Technology, Guangzhou 510640

²Department of Physics, Hong Kong University of Science and Technology, Hong Kong

A three-company econophysics model for competing multi-agent systems in a triangular lattice is analyzed using the mean field theory for its phase diagram. Interpretations for the temperature, spin density and lattice structures are presented. Suggestions of the usage of this model for econophysics in the context of multi-agent systems are made.

PACS: 89.20.-a, 89.65.Gh, 05.70.Fh

Competition of chain stores in a city is an interesting problem in which the tools of statistical analysis within the framework of multi-agent systems can be employed. In recent papers, competition and collaborations among companies in a two-dimensional point pattern have been described by a soap froth model,^[1,2] where the evolving topology simulates the evolution of agents in the system. In this paradigm, each point representing a particular shopping mall defines a Voronoi cell and the point pattern of shopping mall in a city map becomes a cellular pattern of regions of loyal customers. To introduce the idea with a more concrete example, consider the city map of Hong Kong with many shopping malls and the competing companies are supermarket chain owners. The competition for market share can thus be modeled by the competition for occupancy of the shopping mall. To further develop this mathematical framework, let us assume that there are only two large chains of supermarkets (B and W). If we assume that the center of each cell denotes a shopping mall accommodating one supermarket, then we color the cell white if the supermarket belongs to chain W and black if it belongs to chain B. This cell coloring endows the cellular pattern with an interpretation of the distribution of market share of two supermarket chains. Since the separation between any points within the cell to the center of that cell is shorter than the distance between this point to the center of other cells, customers prefer to shop in the cell in which they reside. On the other hand, those customers living on the boundary between two cells can choose one cell in which they shop, depending on the marketing strategy of the two companies. Thus, two malls with different colors will compete for these customers, and we can expect strong interaction between these neighboring cells.^[3-5] Mathematically, each cell i is initially assigned one of the N colors for N competing companies. In the simplest case, where $N = 2$, we have an economic situation similar to the Ising model in statistical mechanics.^[1] In this Letter, we generalize our earlier work to the case of $N = 3$. We may call it a Potts generalization,

though this name is not really adequate because our econophysics model is more complex. In order to describe N competing companies in market dominance for the shopping malls, we introduce our model for the interaction of agents based on a simple idea: collaborate when two cells are the same color and compete otherwise. In this way, we have a framework for associating the location problem of competing companies in a city with the statistical mechanics of N color on a two-dimensional cellular pattern.

Let us look now at the realistic problem of deciding the locations for the branches of a company. Our first task is to initialize the distribution of the supermarkets belonging to two companies, with one branch per mall. This task is equivalent to a color assignment to the cells in the cellular pattern as the dual to the point pattern of shopping malls. We assume that the two chains contain almost the same resources, so that initially about half of the cells are black. The probability of switching color is then the probability of switching from one state to another in a canonical ensemble, which depends on the Boltzmann factor $e^{-\beta\Delta E}$, where β is related to noise and ΔE is the energy difference between two states. The uncertainty of a particular agent changing loyalty from one company to another can be interpreted as thermal noise in physics. A large β corresponds to low noise or high loyalty. The interaction-energy difference ΔE has many possible interpretations. The first one concerns the customers' choice. Since competing companies will introduce various promotional offers to attract customers, the customers can measure these different offers by ΔE . A second interpretation of ΔE can be made in the context of operating agents: the benefits provided to agent by competing companies to switch loyalty. In the present study, we focus on the static equilibrium property of the pattern. Using the first interpretation, we can say that the Boltzmann factor gives the probability of changing loyalty of the customers and the problem of minimizing energy in physics corresponds to the maximization of benefits for customers. If we use the second interpretation, we

*Supported by CERG under Grant Nos 602506 and 602507.

**Email: phxjfu@scut.edu.cn

© 2009 Chinese Physical Society and IOP Publishing Ltd

can say that the Boltzmann factor gives the probability of changing loyalty of the agents, who are assumed selfish with the simple objective of increasing the profit percentage sharing by a switch of color. In either interpretation, the competing companies aim at achieving dominance in market share by maximizing the benefits to customers or agents. Our aim here is not to expound on the relation of local choices of supermarket to customers satisfaction, or to individual agent's economic consideration. We wish to use the switching dynamics to lower the energy. When energy is decreased, the probability of switching color is unity. In contrast, if energy is increased after switching, there should be a finite probability $e^{-\beta\Delta E}$ that the switching is still active, due to noise.

A generalization of the above two-company model is to consider several competing companies with finite resource, and one naturally considers the Potts model, though as mentioned above this is a misnomer. Let us specialize on a model where there are three companies competing to open up customer service shop in a given shopping mall. Since we like to address the challenging problem of finite resource, we consider the effect of global constraints on resources for the possible equilibrium state of the three companies. This is inspired by the need for a good strategy to allocate chain stores in shopping malls with limited resources for a particular business. To be more specific, let us consider a city with N malls, and each mall with 3 shops leased to three competing companies so that there are $3N$ possible shops. Company A has a finite budget to open a maximum number M_A of shops in selected malls. One realization of this constraint is $M_A = \sum_{i=1}^N x_i^A$, where x_i^A is the number of shops in the i th mall rented by company A. Since it seems unwise for a company to open up two shops in the same mall, we assume that x_i^A can be either 0 or 1. Note that the values of the vector $\{x_i^A\}$ corresponds to a particular strategy of allocation of resources for company A. Similar consideration applies to companies B and C. In order to make the problem interesting, we can assume that no one company can dominate initially by renting all the shops in all the malls, in another words, $0 < M_A, M_B, M_C < N$. Since the total number of shops rented cannot be greater than the total number of shops, we have $M_A + M_B + M_C \leq 3N$, and we can discuss the effect of saturation of all shops by considering the phase diagram in terms of the fraction of all shops rented by all companies, $\rho = (M_A + M_B + M_C)/3N$.

Next, we consider the coupling of agents using the magnetic Hamiltonian

$$H = \sum_{u=A}^C \sum_{v=A}^C J_{ij}^{uv} x_i^u x_j^v, \quad (1)$$

$$J_{ij}^{uv} = \begin{cases} 0 & \text{if } i = j, \\ J_1 & \text{if } i \neq j, u = v, \\ J_2 & \text{if } i \neq j, u \neq v, \end{cases}$$

where J_{ij}^{uv} is the interaction between two agents, one is located at site i and belongs to company u the other is located at site j and belongs to company v . The summation is over pairs of nearest neighbors defined by triangular lattice. Each spin can be in one of the three possible states, $S = 1, 2, 3$. The occupation number of a spin at a given position $\mathbf{r}$ is $n_S(\mathbf{r})$ which can be either 0 or 1, so that $0 \leq \sum_{S=1}^3 n_S(\mathbf{r}) = m(\mathbf{r}) \leq 3$.

Since the companies are competing, we adopt antiferromagnetic interaction between neighboring spins. We will use standard mean field analysis to study the equilibrium states of this generalized antiferromagnetic model with a modified constraint on the occupancy of the spin state in each site.

If two spins at sites i and j are of spin state S_i and S_j , then their interaction is given by

$$J_{S_i S_j} = (J_1 - J_2)\delta_{S_i S_j} + J_2 = \begin{cases} J_1 & \text{for } S_i = S_j, \\ J_2 & \text{for } S_i \neq S_j. \end{cases}$$

When $J_1 > 0$ (< 0), we have antiferromagnetic (ferromagnetic) interaction between two neighboring spins of the same type. When the two neighboring spins are of different type, their interaction is always assumed to be antiferromagnetic with strength $J_2 > 0$.

Fig. 1. A triangular lattice divided into three sublattices.

We need to allocate a given fixed number N_S of spins of state S among these N sites, $N_S \equiv \sum_{\mathbf{r}} n_S(\mathbf{r}) < N$. The occupation probability of spin S is $\rho_S = N_S/N$. Following the recent approach with the mean field method,^[6,7] we divide the triangular lattice into three sublattices as shown in Fig. 1 and assume that the fluctuation of the spin density is small on each sublattice. Let $N_{S\mu}$ be the total number of spins of type S that are in the sublattice μ ($\mu = a, b, c$) subject to the constraint: $\sum_{\mu=a}^c N_{S\mu} = N_S$. The concentration $C_{S\mu}$ of spin S on sublattice μ is given by $C_{S\mu} = N_{S\mu}/(N/3) \leq 1$. The total number of spins on the lattice is $M \equiv \sum_{S=1}^3 N_S = \sum_{S=1}^3 \sum_{\mu=a}^c N_{S\mu}$. Since each site can accommodate up to three spins, the maximum number of spins is $M_{\max} = 3N$. The total occupation probability of all the spins is $\rho = M/3N = (1/3N) \sum_{S=1}^3 \sum_{\mu=a}^c N_{S\mu} = (1/3) \sum_{S=1}^3 \rho_S \leq 1$. If

we assume no preference in the labeling of Potts spin state $S = 1, 2, 3$, then we can assume without loss of generality that $N_1 = N_2 = N_3$ and $\rho_1 = \rho_2 = \rho_3$.

To perform a mean-field analysis we consider the following variational free energy,^[7] $F = (\langle H_0 \rangle - TS_0)/N$. Here $\langle H_0 \rangle$ and S_0 can be expressed in terms of $C_{S\mu}$,

$$\langle H_0 \rangle = \frac{N}{2} \sum_{\mu=a}^c \sum_{\nu=a}^c \sum_{S_1=1}^3 \sum_{S_2=1}^3 C_{S_1\mu} C_{S_2\nu} \cdot [J_2 + (J_1 - J_2)\delta_{S_1 S_2}](1 - \delta_{\mu\nu}), \quad (2)$$

$$S_0 = -\frac{N}{3} \sum_{S=1}^3 (C_{Sa} \ln C_{Sa} + C_{Sb} \ln C_{Sb} + C_{Sc} \ln C_{Sc}). \quad (3)$$

For given ρ_S , J_1 and J_2 we look for $\{C_{S\mu}\}$ to minimize F , thereby obtain the mean field phase diagram.

Since the labeling of the sublattices is arbitrary, we can assume without loss of generality that the concentrations of each of the three spin states on the same sublattice are equal, i.e.,

$$C_{1\mu} = C_{2\mu} = C_{3\mu}, \quad \mu = a, b, c. \quad (4)$$

However, there can be difference in concentration of a particular spin type in different sublattices, as their neighborhood can be different. Thus, we assume

$$\begin{aligned} C_{Sa} &= x, & C_{Sb} &= y, \\ C_{Sc} &= z = 3\rho - x - y, & S &= 1, 2, 3, \end{aligned} \quad (5)$$

with $0 \leq x, y, z \leq 1$. We now can rewrite the total free energy as

$$F = J(3\rho x + 3\rho y - x^2 - y^2 - xy) + T[x \ln x + y \ln y + (3\rho - x - y) \ln(3\rho - x - y)], \quad (6)$$

where $J = J_1 + 2J_2$.

Fig. 2. Lines of free energy extrema with respect to x and y (in the unit of 3ρ). The intersections locate at the points of minimum free energy.

To locate the free energy minimum, we calculate the first and second derivatives of F and also the discriminant $\Delta = (\partial^2 F / \partial x \partial y)^2 - (\partial^2 F / \partial x^2)(\partial^2 F / \partial y^2)$. We first look for two sets of points (x, y) where $\partial F / \partial x = 0$, $\partial^2 F / \partial x^2 > 0$ and $\partial F / \partial y = 0$, $\partial^2 F / \partial y^2 > 0$, together with the condition $\Delta < 0$. The intersections of these two sets of points give the location of minimum free energy as shown in Fig. 2. We can then discuss two cases.

(1) When, the minimum free energy appears at $A_1 : x = y < \rho$, $z > \rho$, $B_1 : x = z < \rho$, $y > \rho$, and $C_1 : y = z < \rho$, $x > \rho$. Because of the sublattice symmetry, these three solutions represent the same phase.

(2) When $T > T_c$, the minimum free energy appears at. When, the solutions are A_2 , B_2 and C_2 . To confirm the position of the phase transition point, we do further calculations for temperatures near T_c with higher accuracy and the results are shown in the insert in Fig. 2. The solution is at the point E_1 for $T = T_c - \varepsilon$, and E_2 for $T = T_c$. If $T = T_c + \varepsilon$, there is no solution in the range of $x = y < \rho$. The critical temperature can then be identified with the crossing point: for example, we find a critical temperature $T_c \approx 0.86$ for $J_1 = 1$, $J_2 = 0.5$, $\rho = 0.4$.

Fig. 3. The functions $f(x)$ and $g(x)$ and their intersection point determining the critical point.

We now analyze this phase transition in more details. Equation (6) is symmetric for x and y . By setting $x = y$ and defining

$$f(x) = \frac{T}{3J} \ln \frac{x}{3\rho - 2x}, \quad g(x) = x - \rho, \quad (7)$$

we deduce the solutions for $f(x) = g(x)$ as shown in Fig. 3, which locates at the critical point. The critical temperature is determined by $T_c = (3J\rho)/(4 \ln 2) \approx 1.08J\rho$. When $T > T_c$, the system is in a random phase (symmetric phase) and the concentrations of each spin type in each sublattice are equal: $x = y = z = \rho$. When $T < T_c$, the system is in a broken symmetry phase, where each spin type prefers to have its spins occupying one sublattice (c) first, with the remaining spins distributing equally in the two other sublattices: $x = y < \rho$, $z > \rho$. In this broken-

symmetry phase, the concentrations on three sublattices are different, with the concentration in one sublattice (c) to be larger than the other two sublattices (a) and (b).

In conclusion, our mean field analysis using the minimization of the variational free energy provides a good description of the phase diagram for the generalized three state spin model on triangular lattice. Even though the analytical results are quite straightforward, their interpretation is interesting in the context of a generalization of the simple two-company econophysics model. We observe that at low temperature, there is a symmetry broken phase, where a particular spin type will have larger concentration on one sublattice, and smaller but equal concentration on the other two sublattices. Since our model is symmetric in spin types, this observation of the low temperature phase implies that it is advantageous for each company to have its resource allocated more in one sublattice (the symmetry broken phase), with the remaining resource distributed evenly on the remaining sites. Since our mean field analysis does not provide information on the spatial distribution of the shops for each company, we do not know how the three spin types choose their sublattice to allocate more resources. Nevertheless, an efficient way to allocate resource is found at low noise level. Based on our result on the equilibrium states of the model, we anticipate that an interesting extension of the present work will be the dy-

namical assignment of the resources, from which we can see the evolution of the spin concentration on the sublattices.^[2] Of special interest to economic strategist for a given company is that symmetry-broken assignment of resource on the lattice is favorable in a competing world of agents, provided that the noise factor, signified by the thermal energy, is lower than a critical value, $k_B T_c$.^[1] One may also like to investigate the generalization of the present studies to random point patterns which in the Ising case shows similar critical behavior as in a regular lattice.^[8,9]

References

- [1] Szeto K Y and Kong Chiwah 2003 *Comput. Econ.* **22** 163
- [2] Cheung W K and Szeto K Y 2005 *Proceeding of the 8-th International Work-Conference on Artificial Neural Networks* p 75
- [3] Hao J K et al 1997 *Artificial Evolution: Third European Conference* (Nimes, France)
- [4] Lee L C, Nwana H S, Ndumu D T and de Wilde P 1998 *BT Technol. J.* **16** 94
- [5] Binmore 1998 *J. Economic Theory* **80** 257
- [6] Itakura M 1997 *Phys. Rev. B* **55** 48
- [7] Binney J J, Dowrick N J, Fisher A J and Newman M E J 1992 *The Theory of Critical Phenomena* (Oxford: Oxford University) p 164
- [8] Fu X J, Szeto K Y and Cheung W K 2004 *Phys. Rev. E* **70** 056123
- [9] Szeto K Y, Fu X J and Tam W Y 2002 *Phys. Rev. Lett.* **88** 138302